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# MATH 350 : REAL ANALYSIS

## Problem Set 10 – due Friday, December 8th

### INSTRUCTIONS:

Please submit this assignment by 4pm on Friday; you may leave it in the mailbox outside my office.

**(0)** Read Chapters 30–33 (pages 102–111).

**(1)** Give an $\epsilon$-$\delta$ proof that $\lim_{x\to 4} \frac{1}{\sqrt{x}} = \frac{1}{2}$. (No algebra of limits allowed!)

**(2)** Textbook problems **30.5, 30.6, 33.2, 33.3, 33.5**.

**(3)** Consider the function $f : (0,1) \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ \frac{1}{n} & \text{if } x = \frac{a}{n} \text{ in reduced terms.} \end{cases}$$

Does $\lim_{x\to 1/3} f(x)$ exist? Either way, justify your answer with a proof.

**(4)** Give concrete examples to show that the following definitions of $\lim_{x\to a} f(x) = L$ are incorrect.

(a) For all $\delta > 0$, exists $\epsilon > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

(b) For all $\epsilon > 0$, exists $\delta > 0$ such that if $|f(x) - L| < \epsilon$ then $0 < |x - a| < \delta$.

**(5)** Suppose $f : [0,1] \to \mathbb{R}$ is monotone increasing, i.e. that $f(x) \leq f(y)$ whenever $x \leq y$.

(a) Show that for any $a \in (0,1)$, $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist.

(b) Let $\mathcal{D}$ denote the set of all points in $[0,1]$ at which $f$ is discontinuous. Prove that $\mathcal{D}$ is countable.

**(6)** Let $0 \leq \alpha < 1$, and let $f : \mathbb{R} \to \mathbb{R}$ be a function that satisfies $|f(x) - f(y)| \leq \alpha|x - y|$ for all $x, y \in \mathbb{R}$. Pick $a_1 \in \mathbb{R}$, and set $a_{n+1} := f(a_n)$ for all $n \in \mathbb{Z}_{\text{pos}}$. Prove that $(a_n)$ converges. [*You may freely use the geometric series formula without proving it.*]

**(7)** The goal of this problem is to explore how continuous functions affect topological properties of sets. (I won't define precisely what I mean by *topological*, but highly recommend taking a course on topology.) Recall that if $\mathcal{A}$ is a subset of the domain of a function $f$, then $f(\mathcal{A}) := \{f(x) : x \in \mathcal{A}\}$.

(a) If $f$ is continuous on a bounded set $\mathcal{B}$, must $f(\mathcal{B})$ be bounded? Prove or give a counterexample.

(b) If $f$ is continuous on a closed interval $\mathcal{C}$, must $f(\mathcal{C})$ be a closed interval? Prove or give a counterexample.

(c) If $f$ is continuous on an open interval $\mathcal{O}$, must $f(\mathcal{O})$ be an open interval? Prove or give a counterexample.