EXISTENCE OF n^{th} ROOTS

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ABSTRACT. I rewrite the proof of Theorem 7.5 (the existence of n^{th} roots) from the textbook, and try to indicate how you might come up with such a proof yourself.

1. STATEMENT OF THE THEOREM

Recall that in class we proved $\sqrt{2} \in \mathbb{R}$; more precisely, we proved the existence of $\alpha > 0$ such that $\alpha^2 = 2$. Theorem 7.5 in the textbook generalizes these results:

Theorem 1.1 (Theorem 7.5 in J-P)**.** *Given $a > 0$ and $n \in \mathbb{Z}_{pos}$, there exists $b > 0$ such that $b^n = a$.*

If the proof in the book seems obscure to you, that's not a surprise: the authors never explain how they arrived at such a proof. From the standpoint of modeling how to write proofs, this is a correct approach, since the process by which the particular author came up with a proof is quite separate from the proof itself! However, since the goal of this course is teach you to not just understand other people's proofs but to devise your own, I've decided it's worth making visible the scratchwork going on behind the scenes; this will occupy us in the next section. In the final section, inspired by the scratchwork, we'll write a short and self-contained proof.

2. SCRATCHWORK

The easiest way to prove the existence of a solution to an equation is often to make a guess about a solution, and then prove that your guess is, in fact, a solution. Inspired by our work with $\sqrt{2}$, there's a fairly natural guess. Let

$$\mathcal{A} := \{x \geq 0 : x^n \leq a\},$$

and set $\beta := \sup \mathcal{A}$. We wish to show that $\beta^n = a$. We'll follow the same model as in the proof of the existence of $\sqrt{2}$: we'll show that both $\beta^n > a$ and $\beta^n < a$ are impossible. By trichotomy, this will force $\beta^n = a$, and we'll be done!

Suppose $\beta^n < a$. It's natural to guess that increasing β by a tiny amount won't change the fact that the n^{th} power is less the a. More precisely, we make the following guess:

Conjecture 2.1. *If $\beta^n < a$, then there exists $\epsilon > 0$ such that $(\beta + \epsilon)^n < a$.*

Note that if we can prove this conjecture, we're done, since it would imply that $\beta + \epsilon \in \mathcal{A}$, contradicting that β is an upper bound of $\mathcal{A}$.

We've already proved the existence of arbitrarily small real numbers, but the Archimedean Property gives us a particularly nice way to actually construct tiny numbers: take $\frac{1}{M}$ for some huge positive integer M. So, to prove our conjecture we'd like to find some huge integer M such that

$$\left(\beta + \frac{1}{M}\right)^n < a.$$

Expanding the left hand side by the binomial theorem, we see that we're trying to find M such that

$$\beta^n + \binom{n}{1}\frac{1}{M}\beta^{n-1} + \binom{n}{2}\frac{1}{M^2}\beta^{n-2} + \cdots + \binom{n}{n-1}\frac{1}{M^{n-1}}\beta + \frac{1}{M^n} < a.$$

Almost every term on the left involves a division by M, which we're imagining is huge, so we hope to make all those terms tiny if we choose M large enough. We manipulate the inequality to isolate all these terms with M's in them:

$$\binom{n}{1}\frac{1}{M}\beta^{n-1} + \binom{n}{2}\frac{1}{M^2}\beta^{n-2} + \cdots + \binom{n}{n-1}\frac{1}{M^{n-1}}\beta + \frac{1}{M^n} < a - \beta^n.$$

Recall that a, β, and n are all fixed numbers; we're trying to find M huge enough to make this inequality true. Since M is a positive integer, $M \geq 1$, so $M^k \geq M$ for any positive integer k. It follows that

$$\begin{aligned}
&\binom{n}{1}\frac{1}{M}\beta^{n-1} + \binom{n}{2}\frac{1}{M^2}\beta^{n-2} + \cdots + \binom{n}{n-1}\frac{1}{M^{n-1}}\beta + \frac{1}{M^n} \\
&\qquad \leq \binom{n}{1}\frac{1}{M}\beta^{n-1} + \binom{n}{2}\frac{1}{M}\beta^{n-2} + \cdots + \binom{n}{n-1}\frac{1}{M}\beta + \frac{1}{M} \\
&\qquad\qquad \leq \left(\binom{n}{1}\beta^{n-1} + \binom{n}{2}\beta^{n-2} + \cdots + \binom{n}{n-1}\beta + 1\right)\frac{1}{M}.
\end{aligned}$$

The quantity in the big parentheses on the right hand side looks complicated, but it's just some fixed positive number. What we need is

$$\left(\binom{n}{1}\beta^{n-1} + \binom{n}{2}\beta^{n-2} + \cdots + \binom{n}{n-1}\beta + 1\right)\frac{1}{M} < a - \beta^n,$$

or in other words, we need

$$M > \frac{\binom{n}{1}\beta^{n-1} + \binom{n}{2}\beta^{n-2} + \cdots + \binom{n}{n-1}\beta + 1}{a - \beta^n}.$$

This will prove our conjecture, and thus show that β^n *cannot be strictly smaller than* a.

Following a similar thought process, we can also find an M huge enough that $\beta^n > a$ would imply $\left(\beta - \frac{1}{M}\right)^n > a$. But this would imply that $\beta - \frac{1}{M}$ is an upper bound of $\mathcal{A}$, contradicting the minimality of β. I leave this half as an exercise, to check your understanding of the first half of the construction.

With this preliminary scratchwork complete, we now formalize the argument.

3. Formal proof of Theorem 7.5

Step 1: Conjecturing the root.

Let

$$\mathcal{A} := \{x \geq 0 : x^n \leq a\}.$$

We wish to find the supremum of this set, but first we must know its supremum exists! It's clear that $\mathcal{A}$ is nonempty (it contains 0), so it suffices to prove that it's bounded above.

Lemma 3.1. *$1 + a$ is an upper bound of $\mathcal{A}$.*

Proof. It's a good exercise in induction to prove that $(1+a)^n \geq 1 + na$ for any positive integer n and $a \geq 0$. It follows that for any $x \in \mathcal{A}$,

$$(1+a)^n \geq 1 + na \geq 1 + a > a \geq x^n.$$

This implies $1 + a > x$, since both x and $1 + a$ are non-negative. This concludes the proof. □

Since $\mathcal{A}$ is nonempty and bounded above, the completeness axiom guarantees the existence of its supremum, and we set

$$\beta := \sup \mathcal{A}.$$

This is our conjectured n^{th} root of a.

STEP 2: $\beta^n \not< a$.

Suppose $\beta^n < a$, so that $a - \beta^n > 0$. The Archimedean Property implies the existence of $M \in \mathbb{Z}_{\text{pos}}$ such that

$$M > \frac{\binom{n}{1}\beta^{n-1} + \binom{n}{2}\beta^{n-2} + \cdots + \binom{n}{n-1}\beta + 1}{a - \beta^n}.$$

This implies

$$\begin{aligned} a - \beta^n &> \binom{n}{1}\beta^{n-1}\frac{1}{M} + \binom{n}{2}\beta^{n-2}\frac{1}{M} + \cdots + \binom{n}{n-1}\beta\frac{1}{M} + \frac{1}{M} \\ &\geq \binom{n}{1}\beta^{n-1}\frac{1}{M} + \binom{n}{2}\beta^{n-2}\frac{1}{M^2} + \cdots + \binom{n}{n-1}\beta\frac{1}{M^{n-1}} + \frac{1}{M^n} \end{aligned}$$

from which it immediately follows that

$$a > \left(\beta + \frac{1}{M}\right)^n.$$

This implies $\beta + \frac{1}{M} \in \mathcal{A}$, which is impossible since β is an upper bound of $\mathcal{A}$! This concludes the proof of this step.

STEP 3: $\beta^n \not> a$.

Exercise.

Since $\beta^n \not< a$ and $\beta^n \not> a$, trichotomy forces $\beta^n = a$. Finally, since β is an upper bound on $\mathcal{A}$ and $0 \in \mathcal{A}$, we see that $\beta \geq 0$; in fact, since $\beta^n = a \neq 0$, we deduce that $\beta > 0$. We've proved that $\mathbb{R}$ contains a positive n^{th} root of a, as claimed. □

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