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MATH 350 : REAL ANALYSIS

SOLUTION SET 1

Textbook Problems

JP1.1 Let $A := \{1, 2, 3, 4\}$ and $B := \{1, 3, 5, 7, 9\}$, both in the universe $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

(a) $A \cup B = \{1, 2, 3, 4, 5, 7, 9\}$
(b) $A \cap B = \{1, 3\}$
(c) $A \setminus B = \{2, 4\}$
(d) $B \setminus A = \{5, 7, 9\}$
(e) $A^c = \{5, 6, 7, 8, 9, 10\}$
(f) $B^c = \{2, 4, 6, 8, 10\}$

JP1.4 **Claim.** $A \subseteq \emptyset$ if and only if $A = \emptyset$.

Proof. As usual with *if and only if* statements, we prove both directions.

($\Leftarrow$) The empty set is a subset of every set.

($\Rightarrow$) We know $\emptyset \subseteq A$, so if $A \subseteq \emptyset$ as well then $A = \emptyset$. □

JP1.11 **Claim.** If $A \subseteq B$ then $B^c \subseteq A^c$.

Proof 1. We prove the contrapositive of the claim. Suppose $B^c \not\subseteq A^c$. Then there exists $x \in B^c \setminus A^c$, or in other words, $x \notin B$ and $x \in A$. This implies $A \not\subseteq B$, a contradiction, so we conclude that $B^c \subseteq A^c$ as desired. □

Proof 2. Assume $A \subseteq B$, and pick an arbitrary $x \in B^c$. Then $x \notin B$, whence $x \notin A$ (since $x \in A \implies x \in B$). But this implies $x \in A^c$. Since $x \in B^c$ was arbitrary, we deduce $B^c \subseteq A^c$. □

(2) Consider the set $A := \{x : x \text{ is a nonempty set}\}$. Is A an element of itself? Briefly justify your answer.

Clearly A is a set; I claim it's clearly nonempty. For example, the set of all currently enrolled Williams students is nonempty, hence is an element of A, so $A \neq \emptyset$. (If you want to be rigorous, $\{\emptyset\}$ is a nonempty set, hence is an element of A, so A must be nonempty.) We conclude that A is a nonempty set. Since A contains *all* nonempty sets, it must contain A itself: $A \in A$.

Note: $\in$ and $\subseteq$ are quite different. For example, $\emptyset \subseteq A$, but $\emptyset \notin A$. Similarly, $A \subseteq A$ for *any* set A, but it's pretty unusual for $A \in A$.

(3) Let S be the set consisting of all sets that aren't elements of themselves. Carefully explain why this set is problematic. [*This problem illustrates the pitfalls of never carefully defining what a set is!*]

There are two possibilities: either $S \in S$ or $S \notin S$. I claim that neither holds, which is a problem.

Suppose $S \in S$. Since S consists of all sets that *aren't* elements of themselves, this would imply that S cannot be an element of S. Contradiction!

Now suppose instead that $S \notin S$. Then, again by the definition of S, we deduce $S \in S$, a contradiction.

NOTE: S is non-empty, since $\{1\} \in S$. (Check this!) But also, S isn't the set of all sets, since the set A from the previous problem isn't an element of S.

(4) Using only the definition of ordered pair (in particular, without using Theorem 2.2 from the book), prove that the only ordered pair corresponding to the set $\{\{1\}\}$ is $(1, 1)$.

Suppose (a, b) is an ordered pair corresponding to the set $\{\{1\}\}$. Then

$$\{\{a\}, \{a, b\}\} = \{\{1\}\}.$$

In particular, $\{\{a\}, \{a, b\}\} \subseteq \{\{1\}\}$, so $\{a\} \in \{\{1\}\}$, which implies $\{a\} = \{1\}$. This implies $a \in \{1\}$, so $a = 1$. A similar argument shows that $\{1, b\} = \{1\}$, meaning $b \in \{1\}$, which implies $b = 1$. $\square$

NOTE. It is not a valid proof to start with the set $\{\{1\}\}$ and expand it to $\{\{1\}, \{1, 1\}\} = (1, 1)$; this merely shows that $(1, 1)$ is one interpretation of $\{\{1\}\}$, but doesn't prove it's the *only* interpretation!