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MATH 350 : REAL ANALYSIS

SOLUTION SET 2

(1) Book problems 2.1, 2.3, 3.3, 3.5, 3.6, and 3.8.

2.1: Let

$$g = \{(1,2),(2,2),(3,1),(4,4)\}$$
$$f = \{(1,5),(2,7),(3,9),(4,17)\}$$
$$A = \{1,2\}$$

Compute...

(a) The domain of g: $\{1,2,3,4\}$
(b) The range (codomain) of g: $\{1,2,4\}$ (note: sets don't see repetitions or order)
(c) $g(A) = \{2\}$
(d) $g^{-1}(A) = \{1,2,3\}$
(e) $f \circ g = \{(1,7),(2,7),(3,5),(4,17)\}$
(f) $f^{-1} = \{(5,1),(7,2),(9,3),(17,4)\}$

2.3: Let $f : X \to Y$ and $A \subset X$ and $B \subset Y$. Prove...

(a) $f(f^{-1}(B)) \subset B$: Given $x \in f\big(f^{-1}(B)\big)$. Then $x = f(y)$ for some $y \in f^{-1}(B)$, which implies $x = f(y) \in B$. Thus every element of $f\big(f^{-1}(B)\big)$ is an element of B, or in other words, $f\big(f^{-1}(B)\big) \subseteq B$. $\square$

(b) $A \subset f^{-1}(f(A))$: Pick $a \in A$. Then $f(a) \in f(A)$. Since $a \in f^{-1}\big(f(a)\big)$, it follows that $a \in f^{-1}(f(A))$.
Alternate approach. If $x \notin f^{-1}\big(f(A)\big)$, then $f(x) \notin f(A)$. Thus $x \notin A$. It follows by contrapositive that any element of A is an element of $f^{-1}\big(f(A)\big)$, or in other words, $A \subseteq f^{-1}\big(f(A)\big)$. $\square$

3.3: Prove that $-(-x) = x$ for all $x \in \mathbb{R}$.

Main caution: nowhere have we proved that $-x = -1 \cdot x$! (That's proved below.) We did, however, prove that any $x \in \mathbb{R}$ has a *unique* additive inverse...
Let y be the additive inverse of $-x$, so $y = -(-x)$. Then we know that $y + (-x) = 0$ and $(-x) + y = 0$. But note that $x + (-x) = 0$ and $(-x) + x = 0$, so x satisfies the properties of the additive inverse of $-x$. Since additive inverses are unique, we conclude that $y = x$.

3.5: Let $x, y \in \mathbb{R}$. Prove that $xy = 0$ if and only if $x = 0$ or $y = 0$.

($\Rightarrow$) If both x, y are zero we're done, so we may assume one of them is nonzero; WLOG, suppose $x \neq 0$. Then x has a multiplicative inverse, and since 0 multiplied by anything gives 0 (proved in class and in the book), we deduce $y = 0$.
($\Leftarrow$) We proved in class (and in the book) that $x \cdot 0 = 0$ for any $x \in \mathbb{R}$.

3.6: Let $x, y \in \mathbb{R}$. Prove that if $xy = xz$ and $x \neq 0$, then $y = z$.

Multiply by $\frac{1}{x}$, which we know exists since $x \neq 0$.

3.8: Prove that $(-1)x = -x$ for all $x \in \mathbb{R}$.

$-1 \cdot x + x = (-1+1) \cdot x = 0 \cdot x = 0$. This means $-1 \cdot x$ is an additive inverse of x, and we proved x has a *unique* additive inverse, so $(-1)x = -x$.

(2) Given a function $f : A \to B$. Prove the following statements without referring to Theorem 2.6 of the book, since the relevant part of that theorem isn't proved there.

(a) If $X, Y \subseteq B$, then $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$.

Observe that

$$\begin{aligned} a \in f^{-1}(X \cap Y) &\iff f(a) \in X \cap Y \iff f(a) \in X \text{ and } f(a) \in Y \\ &\iff a \in f^{-1}(X) \text{ and } a \in f^{-1}(Y) \\ &\iff a \in f^{-1}(X) \cap f^{-1}(Y) \end{aligned}$$

Reading this chain of implications from left to right implies $f^{-1}(X \cap Y) \subseteq f^{-1}(X) \cap f^{-1}(Y)$. Reading it in reverse implies $f^{-1}(X) \cap f^{-1}(Y) \subseteq f^{-1}(X \cap Y)$. We deduce that the two sets must be equal. $\square$

(b) If $X, Y \subseteq A$, then $f(X \cup Y) = f(X) \cup f(Y)$.

Observe that

$$\begin{aligned} a \in f(X) \cup f(Y) &\iff a \in f(X) \text{ or } f(Y) \\ &\iff a = f(t) \text{ for some } t \in X \cup Y \\ &\iff a \in f(X \cup Y). \end{aligned}$$

Thus the two sets are equal. $\square$

(3) Suppose a set S consists of exactly three (distinct) elements and satisfies (A1)–(A4). Must there exist $x \in S$ such that $x + x \neq 0$? Either prove such an x must exist, or give an example of a set S in which there's no such x.

Proposition. S must contain an element x satisfying $x + x \neq 0$.

Proof 1. Write $S = \{0, a, b\}$, and suppose $x + x = 0$ for every $x \in S$. I claim this implies that the three elements cannot all be distinct, contradicting the hypothesis on S.

By (A1), S is closed under addition, so $a + b \in S$. Thus, exactly one of the following three relations must hold: $a + b = 0$, $a + b = a$, $a + b = b$.

- If $a + b = 0$, then $b = 0 + b = (a + a) + b = a + (a + b) = a + 0 = 0 + a = a$.
- If $a + b = a$, then $b = 0 + b = (a + a) + b = a + (a + b) = a + a = 0$.
- If $a + b = b$, then $a = 0 + a = a + 0 = a + (b + b) = (a + b) + b = b + b = 0$.

In all three cases, we deduce a contradiction to the hypothesis that S consists of three distinct elements. This concludes the proof. $\square$

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Proof 2. Write $S = \{0, a, b\}$. We know $0 + 0 = 0$, and suppose $a + a = 0$ as well. Here's an addition table that summarizes all this information:

$+$	0	a	b
0	0	a	b
a	a	0	?
b	b	?	?

By associativity, $a + (a + b) = (a + a) + b = b$. But we know that $a + 0 = a$ and $a + a = 0$, which forces $a + b = b$. By commutativity, $b + a = b$ as well. Thus, our addition table becomes

$+$	0	a	b
0	0	a	b
a	a	0	b
b	b	b	?

Finally, observe that

$$a + (b + b) = (a + b) + b = b + b.$$

From the addition table, the only element which yields itself when added to a is b, whence $b + b = b$, so we can complete our addition table:

$+$	0	a	b
0	0	a	b
a	a	0	b
b	b	b	b

Since all elements of S are distinct, we conclude that $b + b \neq 0$. $\square$

NOTE: Observe that in the above addition table, $b + b = b$. **This doesn't contradict the uniqueness of the additive identity;** 0 is still the only element such that $0 + x = x$ for *every* $x \in S$.

(4) (Meta-analytic) For each of the following sets and operations, identify all the axioms out of (A1)-(A5) that don't hold. Whenever an axiom fails to be satisfied, give an example illustrating the failure.

(i) The set $\mathbb{Z} \setminus \{0\}$ where $+$ means multiplication (e.g. $3 + 5 := 15$).

(A5) fails, but all the other axioms hold. Note that the additive identity is 1, but 2 has no inverse in $\mathbb{Z} \setminus \{0\}$. (Its inverse would be $\frac{1}{2}$.)

(ii) The set $\mathcal{P}(\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\})$ (the power set) where $+$ means intersection (i.e. $a + b := a \cap b$).

(A5) fails, but all the other axioms hold. Note that the additive identity is $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, so $\{1, 2\}$ has no additive inverse (since $\{1, 2\} \cap S \subseteq \{1, 2\}$ for any S).

(iii) The set of all positive integers, where $+$ is defined by $a + b := \gcd(a, b)$. (Recall that given two positive integers a and b, the *greatest common divisor* of a and b, denoted $\gcd(a, b)$, is the largest positive integer dividing both a and b.)

All the axioms hold except (A4) and (A5). (A4) fails to hold because $\gcd(a, n) \leq a$, so in particular $\gcd(a, n) \neq n$ for any $n > a$ so no integer a can be an identity. Since (A4) fails, (A5) fails automatically (it's not well-posed).

(iv) The set of all positive integers, where $+$ is defined by $a + b := \operatorname{lcm}(a, b)$. (Recall that given two positive integers a and b, the *least common multiple* of a and b, denoted $\operatorname{lcm}(a, b)$, is the smallest positive integer that is a multiple of both a and b.)

All the axioms hold except (A5). Note that the identity is 1, but for any $a > 1$ we have $\operatorname{lcm}(a, n) \geq a > 1$ so no integer $a > 1$ has an inverse.

(v) The set of all non-negative integers (i.e. $\{0, 1, 2, \ldots\}$), where $+$ is defined by $a + b := |a - b|$.

Every axiom is satisfied except (A2), associativity. Indeed, using the definition of $+$ given, we have $2 + 2 + 4 = 4$ or 0, depending on the grouping of the terms.

(5) (Meta-analytic) A person is walking through a moving train from the back to the front. How quickly is the person moving relative to the ground? According to Einstein, the right way to add two velocities v_1, v_2 pointing in the same direction is by the rule

$$v_1 \,@\, v_2 := \frac{v_1 + v_2}{1 + v_1 v_2 / c^2},$$

where c denotes the speed of light. Prove that the interval $I = (-c, c)$ under the binary operation @ satisfies axioms (A1)-(A5). (In particular, in this model nothing can move faster than the speed of light!)

(A1) Closure. Given $x, y \in (-c, c)$, observe that $c - x$ and $c - y$ must both be positive. It follows that

$$c(x + y) < c(x + y) + (c - x)(c - y) = c^2 + xy.$$

Since the right hand side is positive, we deduce that

$$x \,@\, y = \frac{c^2(x + y)}{c^2 + xy} = \frac{c(x + y)}{c^2 + xy} \cdot c < c.$$

A similar argument that starts with the observation that $c + x$ and $c + y$ are positive yields $x \,@\, y > -c$.

(A2) Associativity. A computation shows that

$$x \,@\, y \,@\, z = \frac{c^2(x + y + z) + xyz}{c^2 + xy + xz + yz}$$

and the right hand side is independent of whether x, y are combined first or y, z are.

(A3) Commutativity. Since both addition and multiplication are commutative, the formula $x \,@\, y := \frac{x+y}{1+xy/c^2}$ inherits commutativity as well.

(A4) Identity. We have

$$0 \,@\, x = \frac{0 + x}{1 + x \cdot 0/c^2} = x$$

for any $x \in (-c, c)$.

(A5) Inverses. For any $x \in (-c, c)$, I claim $-x \in (-c, c)$ is its inverse:

$$-x \,@\, x = \frac{x - x}{1 - x^2/c^2} = 0.$$

(6) In Theorem 3.4 of the book, it's shown that $x \cdot 0 = 0$ for all $x \in \mathbb{R}$. Here's an alternative proof:

$$\begin{aligned} x \cdot 0 &= x \cdot (-1+1) \qquad \text{by (A5)} \\ &= x \cdot (-1) + x \cdot 1 \qquad \text{by (A11)} \\ &= -x + x \qquad \text{by (A9)} \\ &= 0 \qquad \text{by (A5).} \qquad \square \end{aligned}$$

There's a very good reason the book didn't give this proof. What is it? (Your answer can be very short, so long as it is compelling.)

In the second step of this "proof" we used that $x \cdot (-1) = -x$, which we proved in **3.8** by using... that $x \cdot 0 = 0$. Thus the proof above is circular!