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MATH 350 : REAL ANALYSIS

Solution Set 3

4.2 Prove parts (i) and (iii) of Theorem 4.5:
(i) Let $\epsilon > 0$. Then $|x| < \epsilon$ if and only if $-\epsilon < x < \epsilon$ and $|x| \leq \epsilon$ if and only if $-\epsilon \leq x \leq \epsilon$.
(iii) $|xy| = |x||y|$ for all $x, y \in \mathbb{R}$.

(i) ($\Longrightarrow$) Suppose $|x| < \epsilon$. By Theorem 4.5 (ii), $|x| \geq x$, so $\epsilon > |x| \geq x$ takes care of $\epsilon > x$. Thus, it suffices to show $x > -\epsilon$, which is equivalent to showing $x + \epsilon > 0$. Either $x > 0, x = 0, x < 0$. If $x > 0$ by closure of positives $x + \epsilon > 0$. If $x = 0$ then $x + \epsilon = \epsilon > 0$. If $x < 0$ then notice $|x| = -x$, which tells us $|x| = -x < \epsilon$. Since $-1 < 0$ ($1 > 0$ so $-1 < 0$ by trichotomy), by theorem 4.2 part v we know

$$\begin{aligned} -x &< \epsilon \\ (-1)(-x) &> (-1)\epsilon \\ x &> -\epsilon, \end{aligned}$$

where $(-1)(-x) = x$ is due to a proposition we proved last problem set along with uniqueness of identity.

($\Longleftarrow$) Suppose $-\epsilon < x < \epsilon$. Either $x \geq 0$ or $x < 0$. If $x \geq 0$ then $|x| = x$, which by our supposition tells us $|x| < \epsilon$. If $x < 0$ (so $|x| = -x$) then $-\epsilon < x$ rearranges to $|x| = -x < \epsilon$.

We can use our work above to simplify the proof that $|x| \leq \epsilon \iff -\epsilon \leq x \leq \epsilon$.
($\Longrightarrow$) If $|x| < \epsilon$ then our work above implies $-\epsilon \leq x \leq \epsilon$. If $|x| = \epsilon$ then either $x = |x| = \epsilon$ or $-x = |x| = \epsilon$. This implies $x = \pm\epsilon$, so in particular, $-\epsilon \leq x \leq \epsilon$.
($\Longleftarrow$) If $-\epsilon < x < \epsilon$, our work above implies $|x| \leq \epsilon$. If $x = \pm\epsilon$, then $|x| = x$ (if $x > 0$) or $-x$ (if $x < 0$); either way, $|x| \leq \epsilon$ in this case.

(iii) Suppose at least one of x, y are 0. WLOG let $x = 0$. Then

$$\begin{aligned} |xy| &= |0 \cdot y| \\ &= |0| \\ &= 0 \\ &= 0 \cdot |y| \\ &= |x| \cdot |y|. \end{aligned}$$

Suppose $x, y \in \mathbb{P}$, i.e. both are positive. Then $xy \in \mathbb{P}$ by closure of positives. Then,

$$|xy| = xy = |x||y|.$$

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Suppose one of x, y are negative. WLOG let $x < 0$, i.e. $|x| = -x$ and $|y| = y$. Also notice that

$$(-x)y = (-1 \cdot x)y = -1(xy) = -xy$$

by what we proved in pset 2. Further $-xy$ is positive by closure. Thus,

$$\begin{aligned} |xy| &= |-1(-x)y| \\ &= |-1(-xy)| \\ &= -xy \\ &= |x||y|. \end{aligned}$$

Finally, suppose both x and y are negative, i.e. $|x| = -x, |y| = -y$. Further, by closure of positives $(-x)(-y)$ is positive. Thus,

$$\begin{aligned} |xy| &= |(-1)(-x) \cdot (-1)(-y)| \\ &= |(-1)(-1)(-x)(-y)|| \\ &= |-(-1)(-x)(-y)|| \\ &= |(-x)(-y)|| \\ &= (-x)(-y) \\ &= |x||y|. \end{aligned}$$

Since these are all the possible cases, we're done.

4.3 Given $x, y \in \mathbb{R}$ such that $x \leq y$ and $y \leq x$. Prove that $x = y$.

If $x \leq y$, then $y - x \in \mathbb{P} \cup \{0\}$. If $x \geq y$, then $x - y \in \mathbb{P} \cup \{0\}$. Further, **3.8** and **3.3** imply

$$-(y - x) = (-1)(y - x) = (-1)y + (-1)(-x) = -y + -(-x) = -y + x = x - y.$$

Putting all the above together, we conclude that if $x \geq y$ and $x \leq y$ then

$$y - x \in \mathbb{P} \cup \{0\} \qquad \text{and} \qquad -(y - x) \in \mathbb{P} \cup \{0\}.$$

If $x \neq y$, the first condition guarantees $y - x \in \mathcal{P}$, while the second condition guarantees $-(y - x) = x - y \in \mathbb{P}$, contradicting trichotomy. Thus $x \neq y$ cannot hold, i.e., $x = y$. $\square$

4.6 Prove that $x^2 > 0$ for all $x \in \mathbb{R} \setminus \{0\}$.

We first prove:

Lemma 1. *For any $x \in \mathbb{R}$ we have $(-x)(-x) = x^2$.*

From this the claim follows almost immediately: if $x \neq 0$, trichotomy implies either $x \in \mathbb{P}$ or $-x \in \mathbb{P}$, and in either case the closure of $\mathbb{P}$ under multiplication yields $x^2 \in \mathbb{P}$. It remains only to prove the Lemma:

Proof of Lemma. We have

$$\begin{aligned} x^2 = x \cdot x = x \cdot \Big(-(-x)\Big) \quad & \text{by } \mathbf{3.3} \\ = x \cdot (-1) \cdot (-x) \quad & \text{by } \mathbf{3.8} \\ = (-1) \cdot x \cdot (-x) \quad & \text{by commutativity and associativity} \\ = (-x)(-x) \quad & \text{by } \mathbf{3.8}. \end{aligned}$$

□

(2) (Meta-analytic) Recall that $\mathbb{C} := \{a + bi : a, b \in \mathbb{R}\}$, the collection of complex numbers. Prove that $\mathbb{C}$ isn't an ordered field, i.e. that it doesn't satisfy (A1)-(A12).

I claim that $\mathbb{C}$ fails to satisfy (A12). Indeed, suppose there existed $\mathbb{P} \subseteq \mathbb{C}$ satisfying (A12). We'll prove that $i \notin \mathbb{P}$, $-i \notin \mathbb{P}$, and $i \neq 0$, violating trichotomy. First observe that $1 \in \mathbb{P}$, since we proved in class that the multiplicative identity of *any* set satisfying (A1)–(A12) must be positive. It follows that $i \notin \mathbb{P}$, since otherwise $-1 = i \cdot i$ would live in $\mathbb{P}$, which we know isn't the case. But the same argument shows that $-i$, the additive inverse of i, can't live in $\mathbb{P}$ either! And since $0 \in \mathbb{C}$ and the additive identity is unique, we see that $i \neq 0$. Thus trichotomy cannot be satisfied, so no set $\mathbb{P} \subseteq \mathbb{C}$ satisfying (A12) can exist.

(3) (Meta-analytic) Let $\mathbb{F}_7 := \{0, 1, 2, 3, 4, 5, 6\}$, endowed with two operations + (mod 7) and $\cdot$ (mod 7). Prove that $\mathbb{F}_7$ isn't an ordered field.

Notice $x + 0 = x \ \forall x \in \mathbb{F}_7$. Thus, 0 is the additive identity. Since $6+1 = 0$, 6 and 1 are additive inverses. Suppose, for the sake of contradiction, there's a $\mathbb{P} \subseteq \mathbb{F}_7$ satisfying (A12). Then by trichotomy exactly one of $1 \in \mathbb{P}$ or $6 \in \mathbb{P}$. If $1 \in \mathbb{P}$ then by closure $1+1+1+1+1+1 = 6 \in \mathbb{P}$, and if $6 \in \mathbb{P}$ then by closure $6+6+6+6+6+6 = 1 \in \mathbb{P}$, which is a contradiction.

(4) Prove that $1 + 1 \neq 0$. Must this still be true if we only required that $\mathbb{R}$ satisfy (A1)–(A11)? Justify your answer.

We proved in class that $1 > 0$, so Theorem 4.2 implies

$$2 = 1 + 1 > 1.$$

Since the right hand side is positive, we deduce $2 > 0$ (again by Theorem 4.2), and trichotomy immediately implies $1+1 \neq 0$. If we remove (A12), however, the situation changes drastically. Indeed, consider the set $\{0, 1\}$ with respect to the operations + and $\cdot$ defined by the tables

+	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

It is straightforward to verify that this satisfies all of (A1)–(A11), and it's evident from the addition table that $1 + 1 = 0$.

(5) There are real numbers between real numbers!

(i) Prove that $x^{-1} > 0$ for all positive x.

If $x > 0$, then $x \neq 0$ by trichotomy, whence (A10) implies the existence of $x^{-1} \in \mathbb{R}$. We now employ trichotomy to prove the claim:

- $x^{-1} \neq 0$. Otherwise, we'd have $1 = x^{-1} \cdot x = 0 \cdot x = 0$, contradicting (A9).
- $x^{-1} \not< 0$. Otherwise, Theorem 4.2 would imply $1 = x^{-1} \cdot x < 0 \cdot x = 0$, contradicting trichotomy (since we proved that $1 > 0$).

Trichotomy yields that $x^{-1} > 0$, as claimed.

(ii) Suppose $a < b$. Prove $\exists x \in \mathbb{R}$ such that $a < x < b$.

The idea is simple enough: the average of a and b is a real number that's strictly between them. Proving this rigorously is a bit more challenging, however. In particular,

every time there's a multiplication in an inequality, you have to make sure you've already proved that the thing you're multiplying by is positive!

Proposition 1. $a < (a+b) \cdot 2^{-1} < b$.

Proof. Since $a < b$, Theorem 4.2 implies that $a + b < b + b = (1+1)b = 2b$. By the lemma below, we know $2^{-1} > 0$, so (again by Theorem 4.2)

$$(a+b) \cdot 2^{-1} < b.$$

Similarly, we have $a + b > a + a = (1+1)a = 2a$, whence

$$(a+b) \cdot 2^{-1} > a. \qquad \square$$

Lemma 2. $0 < 2^{-1} < 1$.

Proof. We proved in class that $1 > 0$. Theorem 4.2 implies

$$2 = 1 + 1 > 1. \qquad (*)$$

Since the right hand side is positive, we deduce $2 > 0$ (again by Theorem 4.2), whence $2^{-1} > 0$ by the previous part of this problem.

To prove the other half of the claim, we multiply both sides of $(*)$ by 2^{-1}. Because we've already proved 2^{-1} is positive, Theorem 4.2 implies that $1 > 2^{-1}$. $\square$

(6) Suppose $x, y \in \mathbb{R}$ and satisfy the inequality $x \leq y + \epsilon$ for every real number $\epsilon > 0$. Prove that $x \leq y$.

Suppose $x > y$. Then $x - y > 0$, so problem **(4)** would imply the existence of $\epsilon \in \mathbb{R}$ such that $0 < \epsilon < x - y$. But then we would have $x > y + \epsilon$, contradicting the hypothesis.