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MATH 350 : REAL ANALYSIS

Solution Set 4

5.1 Let X be a set of real numbers with least upper bound a. Prove that if $\epsilon > 0$, there exists $x \in X$ such that $a - \epsilon < x \leq a$.

By Problem Set 3 **(5)**, we know there exists a real number $x \in (a - \epsilon, a)$. Notice this isn't enough, as we must find some $y \in X \cap (a - \epsilon, a]$. Since a is the *least* upper bound of X, x isn't an upper bound on X, whence $\exists y \in X$ such that $y > x$. On the other hand, $y \leq a$, since a is an upper bound of X. We conclude that $a - \epsilon < x < y \leq a$, so we've found an element $y \in X \cap (a - \epsilon, a]$.

5.2 Prove that the greatest lower bound of a set of real numbers is unique.

Suppose both α and α' are greatest lower bounds of $A \subseteq \mathbb{R}$. Then both α and α' are lower bounds of A. Since α is a greatest lower bound, $\alpha \geq \alpha'$. But α' is also a greatest lower bound, so $\alpha' \geq \alpha$. By problem **4.3** from the last problem set, $\alpha = \alpha'$.

5.3 Give another proof of Theorem 5.4 (*a nonempty set of real numbers that's bounded below has a greatest lower bound*) by considering the least upper bound of the set $-X := \{-x : x \in X\}$.

Suppose $X \subseteq \mathbb{R}$ is nonempty and bounded below, say, by $a \in \mathbb{R}$. This means, by definition, that $a \leq x$ for every $x \in X$. But then $-a \geq -x$ for all $x \in X$, whence $-a$ is an upper bound on $-X$. Since $-X$ is clearly nonempty, the completeness axiom (A13) implies $\sup -X$ exists; in other words, $\exists \alpha \in \mathbb{R}$ such that
(i) $\alpha \geq -x$ for all $x \in X$, and
(ii) $\alpha \leq \beta$ for any upper bound β of $-X$.
I claim $-\alpha = \inf X$. To see this, first observe that **(i)** implies $-\alpha \leq x$ for all $x \in X$, whence

$$\boxed{-\alpha \text{ is a lower bound on } X.}$$

Next, given any lower bound y of X we have $-y$ is an upper bound of $-X$ (as proved above), so **(ii)** implies $\alpha \leq -y$; it follows that $-\alpha \geq y$. In other words,

$$\boxed{-\alpha \text{ is at least as large as any lower bound on } X.}$$

The two boxed statements imply that $-\alpha$ is the greatest lower bound of X.

5.5 Let X and Y be nonempty subsets of real numbers such that $X \subset Y$ and Y is bounded above. Prove that $\sup X \leq \sup Y$.

The fact that $X \subseteq Y$ has two immediate consequences:

- any upper bound of Y is automatically an upper bound of X, and
- $X \neq \emptyset$ implies $Y \neq \emptyset$.

Thus both X and Y are non-empty and bounded above, so the completeness axiom (A13) guarantees both $\sup X$ and $\sup Y$ exist (i.e. are real numbers). Since $\sup Y$ is an upper bound on Y, it must be an upper bound on X; in particular, $\sup Y$ must be at least as large as the *least* upper bound on X, which is $\sup X$. In symbols, $\sup Y \geq \sup X$.

5.6 Let X be the set of real numbers with least upper bound a. Let $t \geq 0$. Prove that ta is the least upper bound of the set $tX := \{tx : x \in X\}$.

The claim is trivial if $t = 0$, so we assume henceforth that $t > 0$.

First we prove that ta is an upper bound of tX: we know a is an upper bound of X, whence $a \geq x$ for all $x \in X$. It follows that $ta \geq tx$ for all $x \in X$, so ta is an upper bound of tX.

Next we prove that ta is the least upper bound of tX. Suppose y is an upper bound of tX, so that $y \geq tx$ for all $x \in X$. Since $t > 0$, we deduce $x \leq y/t$ for all $x \in X$, so y/t is an upper bound on X. Since a is the least upper bound on X, we see that $a \leq y/t$. It follows that $ta \leq y$. Since y was an arbitrarily chosen upper bound, we conclude that ta is the least upper bound of X.

5.7 Let X and Y be sets of real numbers with least upper bounds a and b, respectively. Prove that $a + b$ is the least upper bound of the set $X + Y := \{x + y : x \in X, y \in Y\}$.

Since a and b are upper bounds on X and Y, respectively, we see that $x + y \leq a + b$ for all $x \in X$ and $y \in Y$, whence $a + b$ is an upper bound on $X + Y$.

It remains to show that every upper bound of $X+Y$ is at least as large as $a+b$, or equivalently, that every real number smaller than $a + b$ isn't an upper bound of $X + Y$. Pick an arbitrary $z < a + b$, and set $\epsilon := (a + b) - z > 0$. By **(5.1)**, there exist $x \in X \cap (a - \frac{\epsilon}{2}, a]$ and $y \in Y \cap (b - \frac{\epsilon}{2}, b]$, whence

$$x + y > a + b - \epsilon = z$$

so z isn't an upper bound of $X + Y$. It follows that $a + b$ is the least upper bound of $X + Y$.

6.2 Prove that if $m, n \in \mathbb{Z}_{\text{pos}}$, then $mn \in \mathbb{Z}_{\text{pos}}$.

We prove this by induction. Let $S(n)$ denote the assertion "$mn \in \mathbb{Z}_{\text{pos}}$ for every $m \in \mathbb{Z}_{\text{pos}}$". Clearly $S(1)$ holds.

Suppose $S(n)$ holds. Then for any $m \in \mathbb{Z}_{\text{pos}}$ we have $m(n + 1) = mn + m$ by distributive and $mn \in \mathbb{Z}_{\text{pos}}$ by inductive hypothesis. Since we proved in class that $\mathbb{Z}_{\text{pos}}$ is closed under addition, we deduce that $m(n + 1) \in \mathbb{Z}_{\text{pos}}$ for every $m \in \mathbb{Z}_{\text{pos}}$, thus proving that $S(n + 1)$ holds whenever $S(n)$ does. $\square$

6.3 Prove the binomial theorem: for any $a, b \in \mathbb{R}$ and $n \in \mathbb{Z}_{\text{pos}}$, we have

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k,$$

where $\binom{n}{k} := \frac{n!}{k!(n-k)!}$.

We'll require the following result:

Lemma. $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$

Proof. We have

$$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} = \frac{\big((n-k+1)+k\big)n!}{k!(n-k+1)!} = \frac{(n+1)!}{k!(n+1-k)!}$$

and the claim immediately follows. □

We now prove the binomial theorem by induction. Let $S(n)$ denote the assertion of the theorem. Clearly $S(1)$ holds. Now suppose $S(n)$ holds. Then

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k \\
&= \sum_{k=0}^{n}\binom{n}{k}a^{n-k+1}b^k + \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k+1} \\
&= a^{n+1} + \sum_{k=1}^{n}\binom{n}{k}a^{n-k+1}b^k + \sum_{k=1}^{n}\binom{n}{k-1}a^{n-k+1}b^k + b^{n+1} \\
&= a^{n+1} + \sum_{k=1}^{n}\left(\binom{n}{k} + \binom{n}{k-1}\right)a^{n-k+1}b^k + b^{n+1} \\
&= a^{n+1} + \sum_{k=1}^{n}\binom{n+1}{k}a^{n-k+1}b^k + b^{n+1} \\
&= \sum_{k=0}^{n+1}\binom{n+1}{k}a^{n+1-k}b^k
\end{aligned}$$

6.4 If $\emptyset \neq X \subseteq \mathbb{Z}_{\text{pos}}$ and X is bounded above, then X contains a greatest element.

Our idea will be to reflect X across 0 and consider $-X$; by translating this set sufficiently far to the right, we get a subset of $\mathbb{Z}_{\text{pos}}$, which must have a least element. This, in turn, will tell us what the greatest element of X must be.

Say X is bounded above by $y \in \mathbb{R}$. By the Archimedean property, there's a positive integer $a > y$. In particular, everything in X is a positive integer strictly between 0 and a, which implies $-X + a \subseteq \mathbb{Z}_{\text{pos}}$. By the well-ordering of $\mathbb{Z}_{\text{pos}}$ there must exist a least element m of the set, i.e. $m \in -X + a$ and $m \leq x$ for all $x \in -X + a$.

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Claim. *$a - m$ is the greatest element of X.*

Proof. First, since $m \in -X + a$, it immediately follows that $a - m \in X$. Thus it suffices to prove maximality of $a - m$. Pick any $t \in X$. We know that $-t + a \geq m$, whence $t \leq a - m$ a claimed. □

6.5 Find the flaw in the 'proof' of the following 'theorem' that asserts, in a complicated form, that any two positive integers are equal.

Theorem 1. *Given $k, m \in \mathbb{Z}_{pos}$. If $n = \max\{k, m\}$, then $n = k = m$.*

Proof. Let $S(n)$ be the statement of the theorem. If $1 = \max\{k, m\}$, then $k = m = 1$ by Theorem 6.5, so $S(1)$ is true. Now suppose $S(n)$ is true. If $n + 1 = \max\{k, m\}$, then $n = \max\{k - 1, m - 1\}$, whence $S(n)$ implies $n = k - 1 = m - 1$. But then $n + 1 = k = m$, and we've shown that $S(n + 1)$ holds. By induction $S(n)$ must hold for all $n \in \mathbb{Z}_{\text{pos}}$. □

This proof is essentially correct! The **only** flaw in the proof is as follows. It's true that if $n + 1 = \max\{k, m\}$, then $n = \max\{k - 1, m - 1\}$. It does not follow, however, that $k - 1$ and $m - 1$ are positive integers, so the inductive hypothesis doesn't apply. More precisely, $S(1)$ doesn't imply $S(2)$.

NOTES.Many of you thought the proof proceeded in the opposite direction from Induction, but this is not the case: it deduces that $S(n + 1)$ is true from the assumption that $S(n)$ is true. (That said, I agree that the proof could have been more clearly written!)

6.6 Prove Strong Induction: Suppose that $S(n)$ is a sequence of assertions, one for each $n \in \mathbb{Z}_{\text{pos}}$. If

(a) $S(1)$ is true, and

(b) $S(n)$ is true whenever $S(k)$ is true for all positive integers $k < n$.

Then $S(n)$ is true for all positive integers n.

We prove this by induction. Let $R(n)$ be the assertion "$S(k)$ is true for all positive integers $k \leq n$". $R(1)$ is true, by the first hypothesis of Strong Induction that $S(1)$ is true. If $R(n)$ is true, then by the second hypothesis of Strong Induction, $S(n + 1)$ is true. But this means that $S(k)$ is true for all positive integers $k \leq n + 1$, whence $R(n + 1)$ is true. By induction, $R(n)$ is true for all positive integers n. This immediately implies $S(n)$ is true for all positive integers n.

(1) A well-known puzzle game works as follows. (See figure below.) There are three vertical posts, numbered 1, 2, and 3. At the start of the game, there's a stack of n concentric rings of decreasing radius stacked on post 1, and the other two posts are empty. The goal is to end up with the entire stack of rings on post 3, again in order of decreasing radius. A move consists of transporting a ring at the top of any stack to another post; however, no ring can be placed on top of a smaller ring. Prove that the entire stack of n rings can be moved onto post 3 in $2^n - 1$ moves, and that this cannot be done in fewer than $2^n - 1$ moves.

If $2^k - 1$ moves is the minimum possible for k rings, then $2^{k+1} - 1$ is the minimum for $k + 1$ rings, since the bottom ring can't be moved at all until the top k rings are moved somewhere, taking at least $2^k - 1$ moves, the bottom ring has to be moved to spindle 3, taking at least 1 move, and then the other rings have to be placed on top of it, taking at least another $2^k - 1$ moves. Clearly if $n = 1$ it takes only one move, which is $2^n - 1$, so that is our base case and the claim holds by induction.