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MATH 350 : REAL ANALYSIS

Solution Set 6

(1) Suppose $f : A \hookrightarrow B$. Prove that $A \approx f(A)$.

Consider $g : A \to f(A)$ defined $g(a) := f(a)$. g inherits injectivity from f, and g is surjective by definition of $f(A)$. Thus g is a bijection between A and $f(A)$.

(2) Find an explicit bijection $(-1, 1) \hookrightarrow\!\!\!\!\rightarrow \mathbb{R}$. [*Meta-analytic, but don't use functions we haven't defined in class.*]

There are many solutions: any function defined on $(-1, 1)$ whose graph is strictly increasing throughout the interval and has asymptotes at ± 1 will do. Here's one example that's easier to prove things about analytically. One example: let $f : (-1, 1) \to \mathbb{R}$ defined by

$$f(x) := \frac{x}{1 - x^2}.$$

Of course, asserting that it looks bijective is nice, but the pudding is in the proof.

f is injective. If $f(x) = f(y)$, then a bit of algebra implies

$$(1 + xy)(x - y) = 0.$$

Since $xy \in (-1, 1)$, we see that $1 + xy \neq 0$, whence $x = y$.

f is surjective. Pick $y \in \mathbb{R}$; I claim there exists $\alpha \in (-1, 1)$ such that $f(\alpha) = y$. If $y = 0$ the claim is trivial, so we assume henceforth that $y \neq 0$. Let

$$\alpha_1 := -\frac{1}{2y} + \frac{1}{2y}\sqrt{4y^2 + 1} \qquad \text{and} \qquad \alpha_2 := -\frac{1}{2y} - \frac{1}{2y}\sqrt{4y^2 + 1}$$

An easy computation shows that

$$\alpha_1 + \alpha_2 = -\frac{1}{y} \qquad \text{and} \qquad \alpha_1\alpha_2 = -1.$$

Note right away that $|\alpha_1| \neq 1$, since this would force $\alpha_1 + \alpha_2 = 0$ which is impossible.

From above we deduce that for any $x \in \mathbb{R}$,

$$(x - \alpha_1)(x - \alpha_2) = x^2 + \frac{x}{y} - 1 = (1 - x^2)\left(\frac{f(x)}{y} - 1\right).$$

In particular, taking $x = \alpha_1$ or α_2 implies that $f(\alpha_1) = f(\alpha_2) = y$. Since $\alpha_1\alpha_2 = -1$ and $|\alpha_1| \neq 1$, one of α_1 or α_2 must be in the interval $(-1, 1)$. We've thus proved the existence of some $\alpha \in (-1, 1)$ such that $f(\alpha) = y$.

COMMENTS. We haven't discussed π or any trigonometric functions in this course, so any proofs involving those are meta-analytic.

(3) Find an explicit bijection $(0,1] \hookrightarrow\!\!\!\!\rightarrow (0,1)$. Your function is allowed to be defined piecewise, so long as you explicitly state where each element of $(0,1]$ gets sent. [*Hint: Where should you send 1?*]

There are many possible solutions to this, but they all rely on the trick of finding a countably infinite subset of $(0,1]$ and suitably shifting it. Here are a few examples of this.

ANSWER 1.

$$f(x) := \begin{cases} \frac{x}{2} & \text{if } 2^k x = 1 \text{ for some integer } k \geq 0, \\ x & \text{otherwise.} \end{cases}$$

ANSWER 2.

$$g(x) := \begin{cases} \frac{1}{n+1} & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{Z}_{\text{pos}} \\ x & \text{otherwise.} \end{cases}$$

ANSWER 3. We know $\mathbb{Q} \cap (0,1)$ is countable, so we can enumerate all the elements in the form $\mathbb{Q} \cap (0,1) = \{q_1, q_2, q_3, \ldots\}$. Now define $q_0 := 1$, and consider

$$h(x) := \begin{cases} q_{n+1} & \text{if } x = q_n \\ x & \text{if } x \notin \mathbb{Q}. \end{cases}$$

(4) The goal of this exercise is to prove a simple case of Cantor-Schröder-Bernstein (see part (c)).

Check out the proof of the full Cantor-Schröder-Bernstein theorem on the course website.

(a) Prove that of all infinite sets, $\mathbb{Z}_{\text{pos}}$ has the smallest size, i.e. that $\mathbb{Z}_{\text{pos}} \hookrightarrow A$ for any infinite set A.

Since A is infinite, it's nonempty, so there exists $a_1 \in A$. Again by the definition of an infinite set, $|A| \neq 1$, so there exists $a_2 \in A \setminus \{a_1\}$. More generally, suppose we've selected distinct elements $a_1, a_2, \ldots, a_n$ of A (where $n \in \mathbb{Z}_{\text{pos}}$). Clearly $A \neq \{a_1, a_2, \ldots, a_n\}$, since otherwise the bijection $\{1, 2, \ldots, n\} \hookrightarrow\!\!\!\!\rightarrow A$ given by $k \mapsto a_k$ would mean that A is finite. It follows that $A \setminus \{a_1, a_2, \ldots, a_n\} \neq \emptyset$, so we can choose $a_{n+1} \in A \setminus \{a_1, a_2, \ldots, a_n\}$, and the process continues. Note that for any $k < \ell$ we have $a_k \neq a_\ell$.

Now consider the map $\mathbb{Z}_{\text{pos}} \to A$ defined by $n \mapsto a_n$. To prove this is an injection, we must show that distinct inputs map to distinct outputs. Pick $k \neq \ell$; without loss of generality, $k < \ell$. From above we know that $a_k \neq a_\ell$. The map is an injection!

(b) Suppose $A \hookrightarrow \mathbb{Z}_{\text{pos}}$. Without using Cantor-Schröder-Bernstein, prove that A must be countable.

If A is finite, we're done, so we henceforth assume A is infinite.

We're given the existence of some map $f : A \hookrightarrow \mathbb{Z}_{\text{pos}}$. By **(1)**, $A \approx f(A)$. Since $f(A) \subseteq \mathbb{Z}_{\text{pos}}$, Theorem 9.1 from the book implies $f(A)$ is countable; since A is infinite and $A \approx f(A)$, we see that $f(A)$ is infinite. Thus $f(A)$ is infinite and countable, i.e. $f(A) \approx \mathbb{Z}_{\text{pos}}$. We've proved that $A \approx f(A) \approx \mathbb{Z}_{\text{pos}}$, which shows that A is countable.

(c) Prove (without Cantor-Schröder-Bernstein) that if $A \hookrightarrow \mathbb{Z}_{\text{pos}}$ and $\mathbb{Z}_{\text{pos}} \hookrightarrow A$ then $A \approx \mathbb{Z}_{\text{pos}}$.

We're given the existence of some injection $f : \mathbb{Z}_{\text{pos}} \hookrightarrow A$. In particular, $\mathbb{Z}_{\text{pos}} \approx f(\mathbb{Z}_{\text{pos}}) \subseteq A$, which means A must be infinite. On the other hand, part (b) implies that A must be countable. Thus, by definition, $A \approx \mathbb{Z}_{\text{pos}}$.

(5) In class we sketched an argument for $\mathbb{Q}_{\text{pos}}$ being countable. Here we give a rigorous proof of this.

(a) Prove that if $A \hookrightarrow B$ and $B \hookrightarrow C$ then $A \hookrightarrow C$. [*Colloquially: if B is at least as large as A, and C is at least as large as B, then C is at least as large as A.*]

Say $f : A \hookrightarrow B$ and $g : B \hookrightarrow C$. I claim that $g \circ f : A \hookrightarrow C$. Indeed, if $x \neq y$ then $f(x) \neq f(y)$ since f is injective, and then $g\big(f(x)\big) \neq g\big(f(y)\big)$ since g is injective.

(b) Prove that any positive integer can be written in the form $2^k n$, where $k \in \mathbb{Z}_{\text{pos}} \cup \{0\}$ and n is a positive odd integer.

Recall that a positive integer is *even* iff it is an element of $2\mathbb{Z}_{\text{pos}}$; otherwise, it's *odd*. Note that, by this definition, every positive integer is even or odd, but not both. We will need the following

Lemma. *If k is an odd positive integer, then $\exists n \in \mathbb{Z}_{pos}$ such that $k = 2n - 1$.*

We'll prove this below. But first, we use it to solve the problem.

Given $a \in \mathbb{Z}_{\text{pos}}$. If a is odd, we're done: $a = 2^0 a$. If a is even, consider the set

$$S := \left\{ \frac{a}{2^j} : j \in \mathbb{Z}_{\text{pos}} \text{ and } \frac{a}{2^j} \in \mathbb{Z}_{\text{pos}} \right\}.$$

$S \neq \emptyset$ since a is even, hence has a least element, say $n := \frac{a}{2^\ell}$. I claim that n is odd: if not, then $\frac{n}{2} = \frac{a}{2^{\ell+1}}$ would belong to S, contradicting the minimality of n. We deduce that $a = 2^\ell n$ in this case as well.

Proof of Lemma. Suppose k is odd. By definition, this means $k \in \mathbb{Z}_{\text{pos}} \setminus 2\mathbb{Z}_{\text{pos}}$. Consider the set E of all even integers larger than k. Since $\mathbb{Z}_{\text{pos}}$ is well-ordered, E has a least element, e. By construction, $e - k$ is a positive integer strictly smaller than 2. This implies $e - k = 1$, whence $k = e - 1$. Since $e \in 2\mathbb{Z}_{\text{pos}}$, we can write $e = 2n$ for some positive integer n. This concludes the proof. □

(c) Use part (b) to give an explicit bijection $\mathbb{Z}_{\text{pos}} \times \mathbb{Z}_{\text{pos}} \hookrightarrow\!\!\!\!\to \mathbb{Z}_{\text{pos}}$. Prove your map is a bijection.

Consider the map $f : \mathbb{Z}_{\text{pos}} \times \mathbb{Z}_{\text{pos}} \to \mathbb{Z}_{\text{pos}}$ defined by $f(m, n) := 2^{m-1}(2n - 1)$. By part (b), f is a surjection. I claim f is also an injection, which will conclude the proof.

Suppose $f(a, b) = f(m, n)$. Without loss of generality, say $a \geq m$. Then $2^{a-1}(2b - 1) = 2^{m-1}(2n - 1)$, whence

$$2^{a-m}(2b - 1) = 2n - 1.$$

The right hand side is odd, so the left hand side must be as well, whence $a \leq m$. This forces $a = m$, whence $2b - 1 = 2n - 1$, which implies $b = n$. This concludes the proof of injectivity, and it follows that f must be bijective.

(d) Construct an explicit injection $\mathbb{Q}_{\text{pos}} \hookrightarrow \mathbb{Z}_{\text{pos}} \times \mathbb{Z}_{\text{pos}}$.

The natural thing to do is to define a map $\mathbb{Q}_{\text{pos}} \to \mathbb{Z}_{\text{pos}} \times \mathbb{Z}_{\text{pos}}$ via $\frac{a}{b} \mapsto (a, b)$. However, we have to exercise a bit of care, since **this isn't a function**: under this mapping, $1/2 \mapsto (1,2)$ and $1/2 \mapsto (2,4)$. To get around this, given $q \in \mathbb{Q}_{\text{pos}}$, consider

$$S_q := \{b \in \mathbb{Z}_{\text{pos}} : bq \in \mathbb{Z}_{\text{pos}}\}.$$

This set is nonempty by definition of $\mathbb{Q}_{\text{pos}}$, so well-ordering implies the existence of a minimal element $b_q := \min S_q$. Now we can define the map $g : \mathbb{Q}_{\text{pos}} \to \mathbb{Z}_{\text{pos}} \times \mathbb{Z}_{\text{pos}}$ by $g(q) := (b_q, qb_q)$. I know claim that g is injective. If $g(q) = g(r)$, then $(b_q, qb_q) = (b_r, rb_r)$, which implies $b_q = b_r$ and $qb_q = rb_r$; combining these yields $q = r$.

(e) Combine parts (a), (c), and (d) to give a short, rigorous proof that $\mathbb{Q}_{\text{pos}}$ is countable.

We just proved that $\mathbb{Q}_{\text{pos}} \hookrightarrow \mathbb{Z}_{\text{pos}} \times \mathbb{Z}_{\text{pos}}$, and in part (c) we proved $\mathbb{Z}_{\text{pos}} \times \mathbb{Z}_{\text{pos}} \hookrightarrow \mathbb{Z}_{\text{pos}}$, so part (a) implies $\mathbb{Q}_{\text{pos}} \hookrightarrow \mathbb{Z}_{\text{pos}}$. On the other hand, the identity map injects $\mathbb{Z}_{\text{pos}}$ into $\mathbb{Q}_{\text{pos}}$. Problem **(4c)** implies that $\mathbb{Q}_{\text{pos}}$ must be countable.

(6) Given a set A, let $\mathcal{F}$ denote the set of all functions $f : A \to \{0,1\}$. Prove that $\mathcal{F} \approx \mathcal{P}(A)$, and use this to explain why some use the notation 2^A rather than $\mathcal{P}(A)$.

Consider the map $\varphi : \mathcal{F} \to \mathcal{P}(A)$ defined by

$$\varphi(f) := \{a \in A : f(a) = 1\}.$$

I claim φ is a bijection.

φ is injective. Pick two distinct elements $f, g \in \mathcal{F}$. Then there must exist some $x \in A$ such that $f(x) \neq g(x)$; WLOG $f(x) = 1$ and $g(x) = 0$. Then $x \in \varphi(f) \setminus \varphi(g)$, which means $\varphi(f) \neq \varphi(g)$.

φ is surjective. Pick $S \in \mathcal{P}(A)$. Let $\chi_S : A \to \{0,1\}$ be the *characteristic function of S*, i.e.

$$\chi_S(x) := \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise.} \end{cases}$$

Then $\varphi(\chi_S) = S$.

We've thus proved φ is a bijection, whence $\mathcal{F} \approx \mathcal{P}(A)$. Note that any element $f \in \mathcal{F}$ assigns one of two values (0 or 1) to each element of A; if A were finite, this would imply there are $2^{|A|}$ possible choices of f.

8.2 Prove that if $X \approx Y$, then $\mathcal{P}(X) \approx \mathcal{P}(Y)$.

Since $X \approx Y$, there exists a bijection $\phi : X \hookrightarrow\!\!\!\!\!\rightarrow Y$. An important observation is

Lemma. *ϕ is invertible, i.e. there exists a function $\psi : Y \to X$ such that $\psi \circ \phi$ is the identity function on X. (We will denote ψ by the symbol ϕ^{-1}.)*

The map ϕ induces a natural map between the power sets:

$$\begin{aligned} \mathcal{P}(X) &\to \mathcal{P}(Y) \\ S &\mapsto \phi(S). \end{aligned}$$

I claim this induced map is a bijection. *continued on next page...*

First, it's an injection: if $\phi(S) = \phi(T)$, then (since ϕ is invertible) $S = T$. It's also a surjection: for any $T \in \mathcal{P}(Y)$, the set $\phi^{-1}(T)$ gets mapped to T. This concludes the proof that $\mathcal{P}(X) \approx \mathcal{P}(Y)$.

Proof of Lemma. By definition of bijection, we know that for each $b \in B$ there exists a unique element $\alpha_b \in A$. Let $\psi(b) := \alpha_b$ for each $b \in B$. By construction, if $\phi(a) = b$ then $\psi \circ \phi(a) = a$. Since this holds for every $a \in A$, ψ is the inverse function of ϕ. □

9.4 If A is uncountable and $A \subseteq B$, prove that B is uncountable.

If B were countable then $B \hookrightarrow \mathbb{Z}_{\text{pos}}$, which would imply $A \hookrightarrow \mathbb{Z}_{\text{pos}}$. By problem **(4b)**, we deduce that A would be countable.

9.3 If A is countable and B is uncountable, prove that $A \cup B$ is uncountable.

This follows from **9.4**, since $B \subseteq A \cup B$.

9.10 Prove that the plane isn't a countable union of lines.

Suppose $\mathcal{F}$ is a countable family of lines in the plane. We'll prove there must be some point in the plane that doesn't belong to any of the lines in $\mathcal{F}$.

Since $\mathbb{R}$ is uncountable, there are uncountably many horizontal lines in the plane. In particular, one of these lines must not belong to $\mathcal{F}$, say, $\mathcal{L}$. Thus each line in $\mathcal{F}$ intersects $\mathcal{L}$ in at most one point. Since $\mathcal{F}$ is countable, while $\mathcal{L}$ has uncountably many points, there must exist a point on $\mathcal{L}$ that isn't on any of the lines in $\mathcal{F}$.

7.9 This will be posted as a separate document.

Challenge Problem (Not for submission, but I'm happy to discuss it with you.)

(9*). If $A \approx [0, 1]$, prove that $A \times A \approx A$. [This is true for any infinite A, but it's much harder to prove for A strictly larger than $[0, 1]$.]

We'll prove the claim for $A = [0, 1]$. It's clear that $[0, 1] \hookrightarrow [0, 1] \times [0, 1]$, for example, via the map $x \mapsto (x, x)$. By Cantor-Schröder-Bernstein, it suffices to find an injection in the other direction.

Given $(x, y) \in [0, 1] \times [0, 1]$, express x and y in binary. There is some ambiguity here, since some numbers admit two binary expansions; if this is the case, pick the one that has infinitely many 1's. Denote these binary expansions as

$$x = (0.x_1x_2x_3\ldots)_2 \qquad \text{and} \qquad y = (0.y_1y_2y_3\ldots)_2$$

where $x_i, y_i \in \{0, 1\}$ for all i. Now consider the *decimal* number

$$\varphi(x, y) := 0.z_1z_2z_3\ldots$$

where $z_i := x_i + y_i$ for every i. It's an exercise to show that $\varphi : [0, 1] \times [0, 1] \hookrightarrow [0, 1]$. (An alternative approach is to interlace the binary digits of x and y.)