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MATH 350 : REAL ANALYSIS

Solution Set 7

(1) Checking some fundamentals...

(a) Is $\infty \in \mathbb{R}$? Carefully formulate what properties you'd like such a number to have, and then prove that it is or isn't an element of $\mathbb{R}$.

Whichever reasonable properties you might require of ∞, it cannot live in $\mathbb{R}$. For example, you might expect $\infty + 1 = \infty$; this would contradict trichotomy, since we know $1 > 0$. Or you might wish $\infty > x$ for all $x \in \mathbb{R}$; this would contradict the Archimedean property. If you come up with a property you feel ∞ should have that *doesn't* contradict anything we know about $\mathbb{R}$, please let me know!

(b) Suppose $|x| \leq \epsilon$ for every $\epsilon > 0$. Prove that $x = 0$.

Suppose $|x| \leq \epsilon$ for every positive ϵ. In particular, $|x| \leq \frac{1}{n}$ for every $n \in \mathbb{Z}_{\text{pos}}$. If $x \neq 0$, then we'd have $n \leq \frac{1}{|x|}$ for every $n \in \mathbb{Z}_{\text{pos}}$, contradicting the Archimedean property.

(c) Use (b) to prove that $0.\overline{9} = 1$, where $0.\overline{9}$ denotes the number $0.9999\cdots$ written in decimal notation.

Set $x := 1 - 0.\overline{9}$. Clearly $x \geq 0$, so $|x| = x$. Given $\epsilon > 0$, Archimedean Property yields a positive integer $n > \frac{1}{\epsilon}$. Induction implies $10^n > n$, from which it follows that

$$|x| = x < 0.\underbrace{00\cdots0}_{n-1}1 = \frac{1}{10^n} < \frac{1}{n} < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, (b) implies $x = 0$.

(d) Prove that there does not exist a smallest positive real number.

Suppose α were a smallest positive real number. Then $0 < \alpha \leq \epsilon$ for all $\epsilon > 0$. Part (b) implies $\alpha = 0$, a contradiction.

10.3 Prove $\lim\limits_{n\to\infty} \frac{1}{n+2} = 0$.

Proof. Fix $\epsilon > 0$. For every $n > \frac{1}{\epsilon}$ we have $\left|\frac{1}{n+2} - 0\right| = \frac{1}{n+2} \leq \frac{1}{n} < \epsilon$. □

Comments. Note that we didn't use the bound $n > \frac{1}{\epsilon - 2}$, because this causes problems if $\epsilon \leq 2$!

10.5 Prove $\lim\limits_{n\to\infty} \frac{n}{n+2} = 1$.

Proof. Fix $\epsilon > 0$. For every $n > \frac{10}{\epsilon}$ we have $\left|\frac{n}{n+2} - 1\right| = \frac{2}{n} < \frac{\epsilon}{5} < \epsilon$. □

Comments. Why $\frac{10}{\epsilon}$? Because in analysis, we're never trying to optimize... we just need a bound that's good enough.

10.7 Prove that $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

Proof. Fix $\epsilon > 0$. Then for all $n > \frac{1}{\epsilon}$ we have $\left|\frac{(-1)^n}{n} - 0\right| = \left|\frac{(-1)^n}{n}\right| = \frac{1}{n} < \epsilon$. □

10.12 Suppose $\lim_{n\to\infty} a_n = L$. Prove that $\lim_{n\to\infty} |a_n| = |L|$.

We first prove a helpful consequence of the triangle inequality:

Lemma 1. $|x - y| \geq \big||x| - |y|\big|$.

Proof. The triangle inequality implies $|x-y|+|y| \geq |x|$, whence $|x-y| \geq |x|-|y|$. Switching the roles of x and y, this implies $|y - x| \geq |y| - |x|$. Since $|x - y| = |y - x|$, we deduce that

$$|x-y| \geq |x| - |y| \quad \text{and} \quad |x-y| \geq |y| - |x|,$$

Since $\big||x| - |y|\big|$ is either $|x| - |y|$ or $|y| - |x|$, we conclude that $|x - y| \geq \big||x| - |y|\big|$. □

Claim. $a_n \to L \implies |a_n| \to |L|$.

Proof. Given $\epsilon > 0$. We know there exists N such that $|a_n - L| < \epsilon$ whenever $n > N$. The lemma implies

$$|a_n - L| \geq \big||a_n| - |L|\big|.$$

Thus $\forall n > N$ we have

$$\big||a_n| - |L|\big| \leq |a_n - L| < \epsilon.$$

□

(2) Consider the sequence

$$a_n := \begin{cases} 1 & \text{if } n = 2^k \text{ for some } k \in \mathbb{Z}_{\text{pos}} \\ \frac{1}{n} & \text{otherwise,} \end{cases}$$

so the sequence begins $1, 1, \frac{1}{3}, 1, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, 1, \frac{1}{9}, \ldots$ Does (a_n) converge? Justify your answer with a proof.

We start with two lemmas that will be useful later.

Lemma 2. *For any N, there exists $m \in \mathbb{Z}_{pos}$ such that $2^m > N$.*

Proof. By induction, $2^k > k$ for all $k \in \mathbb{Z}_{\text{pos}}$. Archimedean property implies the existence of $m > N$. Thus $2^m > N$. □

Lemma 3. *For any N, there exists $m > N$ such that $m \neq 2^k$ for any $k \in \mathbb{Z}_{pos}$.*

Proof. Pick any N. By Lemma 2, there exists $\ell \in \mathbb{Z}_{\text{pos}}$ such that $2^\ell > N$. I claim that $2^\ell + 1$ cannot be a power of 2. Indeed, observe that $2^{\ell+1} - 2^\ell = 2^\ell \geq 2$, whence

$$2^\ell < 2^\ell + 1 < 2^\ell + 2 \leq 2^{\ell+1}.$$

It follows that

$$m \leq \ell \implies 2^m \leq 2^\ell < 2^\ell + 1 \qquad \text{and} \qquad m \geq \ell + 1 \implies 2^m \geq 2^{\ell+1} > 2^\ell + 1.$$

There are no positive integers between ℓ and $\ell + 1$, whence $2^\ell + 1 \neq 2^m$ for any $m \in \mathbb{Z}_{\text{pos}}$. □

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Claim. *(a_n) diverges.*

Proof. Suppose the sequence converged, say, $a_n \to L$. Then there exists N such that

$$|a_n - L| < \frac{1}{100}$$

for all $n > N$. Note that we may safely assume that $N \geq 1$. By Lemma 2 (see below), there exists an integer larger than N that's a power of 2, whence

$$|1 - L| < \frac{1}{100}.$$

By Lemma 3 (see below), there exists some integer $k > N$ that's not a power of 2, whence

$$\left|\frac{1}{k} - L\right| < \frac{1}{100}.$$

Since $k \geq 2$, we have

$$\frac{1}{2} \leq \left|1 - \frac{1}{k}\right| = \left|1 - L + L - \frac{1}{k}\right| \leq |1 - L| + \left|\frac{1}{k} - L\right| < \frac{1}{50}.$$

This contradiction implies that (a_n) cannot converge. □

(3) Suppose (a_n) and (b_n) are convergent sequences and $a_n < b_n$ for all $n \in \mathbb{Z}_{\text{pos}}$. Does it follow that $\lim\limits_{n\to\infty} a_n < \lim\limits_{n\to\infty} b_n$? Either prove this, or provide a counterexample.

No, the limits might be equal. For example,

$$\lim_{n\to\infty}\left(1 - \frac{1}{n}\right) = \lim_{n\to\infty} 1$$

(4) We call a sequence (x_n) *bounded* iff the set $\{x_n : n \in \mathbb{Z}_{\text{pos}}\}$ is bounded. Show (by example) that it's possible to have a bounded sequence (a_n) and a convergent sequence (b_n) such that both $(a_n + b_n)$ and $(a_n b_n)$ diverge.

There are many examples, e.g. $a_n = (-1)^n$ and $b_n = 1$.

(5) Given a sequence (a_n), set $b_n := a_{2n} - a_n$.

(a) Suppose (a_n) converges. Must (b_n) converge? Justify your answer with a proof based on the definition of limit (i.e. without using theorems about limits).

Claim. $\lim\limits_{n\to\infty} b_n = 0$.

Proof. Given $\epsilon > 0$. Since (a_n) converges, say to L, then $|a_n - L| < \frac{\epsilon}{2}$ for all sufficiently large n. It follows that

$$|b_n| = |a_{2n} - a_n| = |a_{2n} - L + L - a_n| \leq |a_{2n} - L| + |L - a_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all sufficiently large n. □

(b) Give an example of (a_n) that diverges such that (b_n) diverges.

Let $a_n := n$, which implies $b_n = n$. It therefore suffices to prove that (a_n) diverges. For any $L \in \mathbb{R}$, the Archimedean property implies the existence of $N > L + 1$, which means that for all $n > N$ we have $|a_n - L| > 1$. Thus a_n cannot converge to L.

(c) Give an example of (a_n) that diverges such that (b_n) converges.

The sequence (a_n) from problem **(2)** does the trick, since

$$b_n = \begin{cases} 0 & \text{if } n = 2^k \\ -\frac{1}{2n} & \text{otherwise.} \end{cases}$$

Thus $b_n \to 0$, since for any $\epsilon > 0$ we have $|b_n| < \epsilon$ for all $n > \frac{1}{\epsilon}$.

(6) Consider the sequence $a_n := n + \frac{(-1)^n}{n}$. Does (a_n) converge? Justify your answer with a proof.

Claim. *(a_n) diverges.*

Proof. Note that $a_n \geq n - 1$ for every $n \in \mathbb{Z}_{\text{pos}}$. For any $L \in \mathbb{R}$, the Archimedean property implies the existence of $N > L + 2$, whence for every $n > N$ we have $a_n \geq n - 1 > N - 1 > L$ so

$$|a_n - L| = a_n - L > (N - 1) - L > 1.$$

Thus a_n cannot converge to L. Since L was arbitrary, we conclude that a_n diverges. □