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MATH 350 : REAL ANALYSIS

Solution Set 8

(1) Determine (with proof) $\lim_{n\to\infty} \sqrt{n+3} - \sqrt{n}$.

We have

$$0 \leq \sqrt{n+3} - \sqrt{n} = \frac{\sqrt{n+3}+\sqrt{n}}{\sqrt{n+3}+\sqrt{n}} \cdot (\sqrt{n+3}-\sqrt{n}) = \frac{3}{\sqrt{n+3}+\sqrt{n}} \leq \frac{3}{\sqrt{n}}$$

A straightforward proof shows that $\frac{3}{\sqrt{n}} \to 0$. Thus by Squeeze Theorem, $\sqrt{n+3} - \sqrt{n} \to 0$.

Notes. Note that to apply the squeeze theorem, you need to explicitly state a sequence that bounds the given one below.

(2) Compute $\lim_{n\to\infty} \sqrt[n]{2}$. Do not use Theorem 16.4. [*Hint. Start by proving that* $\left(1+\frac{1}{n}\right)^n \geq 2$.]

By the binomial theorem, we have

$$\left(1+\frac{1}{n}\right)^n \geq 1 + \binom{n}{1}\frac{1}{n} = 2.$$

Thus, $2^{1/n} \leq 1 + \frac{1}{n}$. On the other hand, $2^{1/n} \geq 1$ for all n, since

$$(2^{1/n}-1)(2^{(n-1)/n} + 2^{(n-2)/n} + \cdots + 2^{1/n} + 1) = 2 - 1 > 0.$$

It follows that $1 \leq 2^{1/n} \leq 1 + \frac{1}{n}$, and we conclude the claim by the squeeze theorem.

(3) Compute $\lim_{n\to\infty} \sqrt[n]{1+\frac{n}{n+1}}$. [*Hint. Squeeze theorem!*]

We have

$$1 \leq \left(1+\frac{n}{n+1}\right)^{1/n} \leq 2^{1/n}.$$

The squeeze theorem implies

$$\lim_{n\to\infty}\left(1+\frac{n}{n+1}\right)^{1/n} = 1.$$

(4) Suppose $a_1 > 1$, and let $a_{n+1} = 2 - 1/a_n$ for each positive integer n. Prove that (a_n) converges, and (rigorously) find its limit.

Claim. $a_n \to 1$

Proof. Our first step is to find a non-recursive formula for a_n:

Lemma. *For any* $n \in \mathbb{Z}_{>0}$ *we have* $\frac{1}{a_n-1} = n - 1 + \frac{1}{a_1-1}$.

(The proof, by induction, is given below.) By the algebra of limits, we deduce

$$\lim_{n\to\infty}(a_n - 1) = \lim_{n\to\infty}\frac{1}{n-1}\left(\frac{a_1-1}{a_1-1+\frac{1}{n-1}}\right) = \left(\lim_{n\to\infty}\frac{1}{n-1}\right)\left(\frac{a_1-1}{a_1-1+\lim\limits_{n\to\infty}\frac{1}{n-1}}\right) = 0.$$

Again from the algebra of limits, we deduce $\lim\limits_{n\to\infty} a_n = 1$. □

Proof of Lemma. We prove this by induction. The base case $n = 1$ is trivial to verify, so it suffices to check the inductive step. We have

$$\frac{1}{a_{n+1}-1} = \frac{1}{1-\frac{1}{a_n}} = \frac{a_n}{a_n-1} = 1 + \frac{1}{a_n-1},$$

so if the claim holds for a_n then it must also hold for a_{n+1}. □

(5) Suppose $0 < a_1 < b_1$ and for each positive integer n define

$$a_{n+1} := \sqrt{a_n b_n} \qquad b_{n+1} := \frac{a_n + b_n}{2}$$

(a) Prove that both sequences (a_n) and (b_n) converge.

In words, a_{n+1} is defined to be the *arithmetic mean* of a_n and b_n, while b_{n+1} is defined to be the *geometric mean* of a_n and b_n. Our first observation is that the geometric mean never exceeds the arithmetic mean:

Lemma. *For any* $x, y > 0$ *we have* $\sqrt{xy} \le \dfrac{x+y}{2}$

Proof. We know that $(x-y)^2 \ge 0$. A bit of algebraic manipulation implies $(x+y)^2 \ge 4xy$, which yields the claim. □

The Lemma implies

(1) $$a_n \le b_n \qquad \forall n.$$

It follows that for every n,

(2) $$a_{n+1} = \sqrt{a_n b_n} \ge a_n \qquad \text{and} \qquad b_{n+1} = \frac{a_n+b_n}{2} \le b_n.$$

Putting equations (1) and (2) together, we conclude that

$$a_1 \le a_2 \le \cdots \le a_n \le b_n \le b_{n-1} \le \cdots \le b_1$$

for any n. This shows a_n is bounded above by b_1 and below by a_1, and that it's monotonically increasing. By the MCT, (a_n) converges. Similarly, (b_n) converges. □

(b) Prove that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

From above we know both sequences converge; say $a_n \to A$ and $b_n \to B$. Since $b_{n+1} = \frac{a_n+b_n}{2}$, taking the limit on both sides and using the algebra of limits implies $B = \frac{A+B}{2}$. So $A = B$.

(6) Consider the following two sequences:

$$(a_n) \quad : \quad \sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}}, \cdots$$

$$(b_n) \quad : \quad \sqrt{2},\ \sqrt{2+\sqrt{2}},\ \sqrt{2+\sqrt{2+\sqrt{2}}}, \cdots$$

More formally, we can define $a_1 = b_1 = \sqrt{2}$ and for each positive integer n set

$$a_{n+1} := \sqrt{2a_n} \qquad \text{and} \qquad b_{n+1} := \sqrt{2+b_n}.$$

Finally, we define a third sequence:

$$c_n := \frac{2^n}{b_1 b_2 \cdots b_n}$$

(a) Prove that (a_n) converges, and make a conjecture about what it converges to. (You don't have to prove your conjecture, but you should try to explain where it comes from.)

Observe that if $0 < x < 2$ then $x < \sqrt{2x} < 2$. By induction, it follows that our sequence (a_n) is monotonically increasing and bounded by 2. By the Monotone Convergence Theorem (MCT), it must converge.

Non-rigorously, we can guess the sequence converges to 2. Here's one way to make this guess: if $\alpha = \sqrt{2\sqrt{2\sqrt{2\sqrt{\cdots}}}}$ then $\alpha = \sqrt{2\alpha}$, whence $\alpha = 2$.

(b) Prove that (b_n) converges, and make a conjecture about what it converges to. (You don't have to prove your conjecture, but you should try to explain where it comes from.)

Observe that

$$1 < x < 2 \quad \implies \quad x < \sqrt{2+x} < 2.$$

By induction, it follows that our sequence (b_n) is monotonically increasing and bounded by 2. By the Monotone Convergence Theorem (MCT), it must converge.

Non-rigorously, we can guess the sequence converges to 2. Here's one way to make this guess: if $\beta = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{\cdots}}}}$ then $\beta = \sqrt{2+\beta}$, whence $\beta = 2$.

(c) Prove that (c_n) converges, and make a conjecture about what it converges to. (You don't have to prove your conjecture, but you should try to explain where it comes from.)

Observe that $c_{n+1} = c_n \cdot \frac{2}{b_{n+1}}$. Since $b_k \le 2$ for all k, we deduce that $c_{n+1} \ge c_n$ for every n, so (c_n) is monotone increasing. If we can show that (c_n) is bounded, the MCT will tell us that (c_n) must converge.

I claim that $c_n \le 2$ for all n. Note that, directly from the definition, finding an upper bound on c_n is equivalent to finding a lower bound on $b_1 b_2 \cdots b_n$. This product is quite complicated, but the analogous product $a_1 a_2 \cdots a_n$ is much easier to work out. This motivates the following approach.

Lemma. *$b_n \ge a_n$ for all n.*

Proof. By induction. The base case $n = 1$ clearly holds. From part (a) we know $a_n \le 2$ for all n, whence $b_{n+1} = \sqrt{2 + b_n} \ge \sqrt{a_n + b_n} \ge \sqrt{2a_n} = a_{n+1}$. □

This allows us to translate the problem from getting a lower bound on $b_1 b_2 \cdots b_n$ to getting a lower bound on $a_1 a_2 \cdots a_n$. A straightforward induction proof produces an exact formula for this product:

Proposition. *We have $a_1 a_2 \cdots a_n = 2^{s(n)}$ where $s(n) := \sum_{k=0}^{n-1} \frac{n-k}{2^{k+1}}$.*

It now suffices to get a lower bound on $s(n)$. Some playing around might lead you to conclude

Lemma. $s(n) = n - 1 + \frac{1}{2^n}$.

Proof. This can be proved directly by some clever applications of the geometric series formula, or by induction. I leave the details as an exercise. □

Putting all our work together, we conclude that

$$c_n = \frac{2^n}{b_1 b_2 \cdots b_n} \le \frac{2^n}{a_1 a_2 \cdots a_n} = 2^{n-s(n)} < 2^{n-(n-1)} = 2.$$

Since (c_n) is monotone increasing and bounded above by 2, the MCT implies (c_n) converges. It turns out $c_n \to \frac{\pi}{2}$! Here's a meta-analytic proof.

Lemma. *For any n, $\sin x = 2^n \left(\sin \frac{x}{2^n}\right)\left(\cos \frac{x}{2}\right)\left(\cos \frac{x}{4}\right)\left(\cos \frac{x}{8}\right)\cdots\left(\cos \frac{x}{2^n}\right)$.*

Proof. This comes from iterating the formula $\sin 2x = 2 \sin x \cos x$. □

Corollary. $\frac{\sin x}{x} = \left(\cos \frac{x}{2}\right)\left(\cos \frac{x}{4}\right)\left(\cos \frac{x}{8}\right)\cdots$

Proof. By playing with the Taylor expansion of $\sin x$, one can prove that $\lim_{\ell \to \infty} \ell \sin \frac{x}{\ell} = x$. □

Plugging in $x = \frac{\pi}{2}$ and repeatedly applying the cosine half-angle formula $\cos \frac{1}{2} x = \sqrt{\frac{1+\cos x}{2}}$ yields the claim.

NOTES. Explicitly writing out the definition of c_n, we deduce a cute formula for π using only the number 2:

$$\pi = 2 \cdot \frac{2}{\sqrt{2}} \cdot \frac{2}{\sqrt{2+\sqrt{2}}} \cdot \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}} \cdots$$

Do other numbers have similar expansions?