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## MATH 350 : REAL ANALYSIS

## Solution Set 9

(1) In class, we outlined a proof of the Cauchy criterion that is similar to that given in Chapter 19 of the book. The biggest difference between our proof and the book's is our appeal to

Lily's conjecture: Any sequence has a monotone subsequence.

(By contrast, the book's proof relies on a clever binary search algorithm.) The goal of this problem is to prove Lily's conjecture. Given a sequence  $(a_n)$ , we call  $a_k$  a peak iff  $a_k \ge a_m$  for all  $m \ge k$ .

- (a) Suppose  $(a_n)$  has infinitely many peaks. Prove that  $(a_n)$  has a monotone subsequence. By definition of peak, the subsequence consisting of the peaks forms a monotonically decreasing sequence.
- (b) Suppose  $(a_n)$  has finitely many peaks. Prove that  $(a_n)$  has a monotone subsequence.

There exists some largest M such that  $a_M$  is a peak. Let  $m_1 = M + 1$ . Since  $a_{m_1}$  isn't a peak, there exists  $m_2 > m_1$  such that  $a_{m_1} < a_{m_2}$ . Since  $a_{m_2}$  isn't a peak, there exists  $m_3 > m_2$  such that  $a_{m_2} < a_{m_3}$ . Since there are no more peaks we can continue this process indefinitely, thus creating a monotonically increasing subsequence  $(a_{m_k})$ .

(2) In our proof of Cauchy's criterion, we asserted (without proof) that any Cauchy sequence is bounded. Prove this.

Given a Cauchy sequence  $(a_n)$ , we know there exists N such that  $|a_m - a_n| < 1$  for all m, n > N. Fix some M > N, and set  $a := a_M$ . Then  $a - 1 < a_n < a + 1$  for all n > N, hence is bounded for all such n. Since the set  $\{a_n : n \le N\}$  is finite, it's bounded as well. Thus the set  $\{a_n : n \le N\} \cup \{a_n : n > N\}$  must be bounded as well.

(3) Suppose  $0 < a_1 < b_1$  and for each positive integer n define

$$a_{n+1} := \sqrt{a_n b_n} \qquad \qquad b_{n+1} := \frac{a_n + b_n}{2}$$

(a) Prove that both sequences  $(a_n)$  and  $(b_n)$  converge.

In words,  $a_{n+1}$  is defined to be the *arithmetic mean* of  $a_n$  and  $b_n$ , while  $b_{n+1}$  is defined to be the *geometric mean* of  $a_n$  and  $b_n$ . Our first observation is that the geometric mean never exceeds the arithmetic mean:

**Lemma 1.** For any x, y > 0 we have  $\sqrt{xy} \le \frac{x+y}{2}$ 

*Proof.* We know that  $(x - y)^2 \ge 0$ . A bit of algebraic manipulation implies  $(x + y)^2 \ge 4xy$ , which yields the claim.

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The Lemma implies

(1)

 $a_n \leq b_n \qquad \forall n.$ 

It follows that for every n,

(2) 
$$a_{n+1} = \sqrt{a_n b_n} \ge a_n$$
 and  $b_{n+1} = \frac{a_n + b_n}{2} \le b_n$ .

Putting equations (1) and (2) together, we conclude that

$$a_1 \le a_2 \le \dots \le a_n \le b_n \le b_{n-1} \le \dots \le b_1$$

for any n. This shows  $a_n$  is bounded above by  $b_1$  and below by  $a_1$ , and that it's monotonically increasing. By the MCT,  $(a_n)$  converges. Similarly,  $(b_n)$  converges.  $\Box$ 

(b) Prove that  $\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n$ .

From above we know both sequences converge; say  $a_n \to A$  and  $b_n \to B$ . Since  $b_{n+1} = \frac{a_n + b_n}{2}$ , taking the limit on both sides and using the algebra of limits implies  $B = \frac{A+B}{2}$ . So A = B.

(4) Let  $0 \le \alpha < 1$ , and let  $f : \mathbb{R} \to \mathbb{R}$  be a function that satisfies  $|f(x) - f(y)| \le \alpha |x - y|$  for all  $x, y \in \mathbb{R}$ . Pick  $a_1 \in \mathbb{R}$ , and set  $a_{n+1} := f(a_n)$  for all  $n \in \mathbb{Z}_{pos}$ . Prove that  $(a_n)$  converges.

Note that

$$|a_{n+1} - a_n| = |f(a_n) - f(a_{n-1})| \le \alpha |a_n - a_{n-1}|;$$

by induction,

$$|a_{n+1} - a_n| \le \alpha^{n-1} |a_2 - a_1|$$

for all n. It follows that whenever  $m \ge n$ ,

$$\begin{aligned} |a_m - a_n| &\leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \dots + |a_{n+1} - a_n| \\ &\leq \alpha^{m-2} |a_2 - a_1| + \alpha^{m-3} |a_2 - a_1| + \dots + \alpha^{n-1} |a_2 - a_1| \\ &\leq (1 + \alpha + \alpha^2 + \dots + \alpha^{m-n-1}) \alpha^{n-1} |a_2 - a_1| \\ &\leq \frac{\alpha^{n-1}}{1 - \alpha} |a_2 - a_1|. \end{aligned}$$

Now fix  $\epsilon > 0$ . Since  $\alpha \in [0, 1)$ ,  $\lim_{k \to \infty} \alpha^k = 0$ , so  $\exists N \in \mathbb{Z}_{pos}$  such that  $\frac{|a_2 - a_1|}{1 - \alpha} \alpha^N < \epsilon$ . Then for all m, n > N we have

$$|a_m - a_n| \le \frac{\alpha^N}{1 - \alpha} |a_2 - a_1| < \epsilon$$

This shows that  $(a_n)$  is Cauchy, hence convergent.

NOTES. Any function satisfying  $|f(x) - f(y)| \leq \alpha |x - y|$  for all  $x, y \in \mathbb{R}$  is called *Lipschitz* continuous; in the special case that  $\alpha \in [0, 1)$ , it's called a contraction.

One nice consequence of our proof is that any contraction f must have a fixed point, i.e. there must exist some  $\kappa$  such that  $f(\kappa) = \kappa$ . To see this, construct  $(a_n)$  as in the problem, and let  $\kappa := \lim_{n \to \infty} a_n$ . Then

$$|f(\kappa) - \kappa| \le |f(\kappa) - f(a_n)| + |f(a_n) - \kappa| \le \alpha |\kappa - a_n| + |a_{n+1} - \kappa| \longrightarrow 0$$

as n gets large. It follows that  $|f(\kappa) - \kappa|$  is an arbitrarily small non-negative number, hence must be 0. This proves that  $f(\kappa) = \kappa$ , so we've not only proved the existence of a fixed point, we've constructed a specific one.

In fact, it turns out the fixed point of f is unique. Can you prove this?

(5) Consider the following two sequences:

$$(a_n)$$
 :  $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \cdots$   
 $(b_n)$  :  $\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2}+\sqrt{2}}, \cdots$ 

More formally, we can define  $a_1 = b_1 = \sqrt{2}$  and for each positive integer n set

$$a_{n+1} := \sqrt{2a_n}$$
 and  $b_{n+1} := \sqrt{2+b_n}$ .

Finally, we define a third sequence:

$$c_n := \frac{2^n}{b_1 b_2 \cdots b_n}$$

(a) Prove that  $(a_n)$  converges, and make a conjecture about what it converges to. (You don't have to prove your conjecture, but you should try to explain where it comes from.)

Observe that if 0 < x < 2 then  $x < \sqrt{2x} < 2$ . By induction, it follows that our sequence  $(a_n)$  is monotonically increasing and bounded by 2. By the Monotone Convergence Theorem (MCT), it must converge.

Non-rigorously, we can guess the sequence converges to 2. Here's one way to make this guess: if  $\alpha = \sqrt{2\sqrt{2\sqrt{2\sqrt{\cdots}}}}$  then  $\alpha = \sqrt{2\alpha}$ , whence  $\alpha = 2$ .

(b) Prove that  $(b_n)$  converges, and make a conjecture about what it converges to. (You don't have to prove your conjecture, but you should try to explain where it comes from.)

Observe that

 $1 < x < 2 \implies x < \sqrt{2+x} < 2.$ 

By induction, it follows that our sequence  $(b_n)$  is monotonically increasing and bounded by 2. By the Monotone Convergence Theorem (MCT), it must converge.

Non-rigorously, we can guess the sequence converges to 2. Here's one way to make this guess: if  $\beta = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{\cdots}}}}$  then  $\beta = \sqrt{2 + \beta}$ , whence  $\beta = 2$ . (c) Prove that  $(c_n)$  converges, and make a conjecture about what it converges to. (You don't have to prove your conjecture, but you should try to explain where it comes from.)

Observe that  $c_{n+1} = c_n \cdot \frac{2}{b_{n+1}}$ . Since  $b_k \leq 2$  for all k, we deduce that  $c_{n+1} \geq c_n$  for every n, so  $(c_n)$  is monotone increasing. If we can show that  $(c_n)$  is bounded, the MCT will tell us that  $(c_n)$  must converge.

I claim that  $c_n \leq 2$  for all n. Note that, directly from the definition, finding an upper bound on  $c_n$  is equivalent to finding a lower bound on  $b_1b_2\cdots b_n$ . This product is quite complicated, but the analogous product  $a_1a_2\cdots a_n$  is much easier to work out. This motivates the following approach.

**Lemma 2.**  $b_n \ge a_n$  for all n.

*Proof.* By induction. The base case n = 1 clearly holds. From part (a) we know  $a_n \leq 2$  for all n, whence  $b_{n+1} = \sqrt{2 + b_n} \geq \sqrt{a_n + b_n} \geq \sqrt{2a_n} = a_{n+1}$ .

This allows us to translate the problem from getting a lower bound on  $b_1b_2\cdots b_n$  to getting a lower bound on  $a_1a_2\cdots a_n$ . A straightforward induction proof produces an exact formula for this product:

**Proposition 1.** We have 
$$a_1 a_2 \cdots a_n = 2^{s(n)}$$
 where  $s(n) := \sum_{k=0}^{n-1} \frac{n-k}{2^{k+1}}$ .

It now suffices to get a lower bound on s(n). Some playing around might lead you to conclude **Lemma 3.**  $s(n) = n - 1 + \frac{1}{2^n}$ .

*Proof.* This can be proved directly by some clever applications of the geometric series formula, or by induction. I leave the details as an exercise.  $\Box$ 

Putting all our work together, we conclude that

$$c_n = \frac{2^n}{b_1 b_2 \cdots b_n} \le \frac{2^n}{a_1 a_2 \cdots a_n} = 2^{n-s(n)} < 2^{n-(n-1)} = 2.$$

Since  $(c_n)$  is monotone increasing and bounded above by 2, the MCT implies  $(c_n)$  converges. It turns out  $c_n \to \frac{\pi}{2}$ ! Here's a meta-analytic proof.

**Lemma 4.** For any n,  $\sin x = 2^n \left( \sin \frac{x}{2^n} \right) \left( \cos \frac{x}{2} \right) \left( \cos \frac{x}{4} \right) \left( \cos \frac{x}{8} \right) \cdots \left( \cos \frac{x}{2^n} \right)$ .

*Proof.* This comes from iterating the formula  $\sin 2x = 2 \sin x \cos x$ .

Corollary 1.  $\frac{\sin x}{x} = (\cos \frac{x}{2}) (\cos \frac{x}{4}) (\cos \frac{x}{8}) \cdots$ 

*Proof.* By playing with the Taylor expansion of  $\sin x$ , one can prove that  $\lim_{\ell \to \infty} \ell \sin \frac{x}{\ell} = x$ .

Plugging in  $x = \frac{\pi}{2}$  and repeatedly applying the cosine half-angle formula  $\cos \frac{1}{2}x = \sqrt{\frac{1+\cos x}{2}}$  yields the claim.

NOTES. Explicitly writing out the definition of  $c_n$ , we deduce a cute formula for  $\pi$  using only the number 2:

$$\tau = 2 \cdot \frac{2}{\sqrt{2}} \cdot \frac{2}{\sqrt{2+\sqrt{2}}} \cdot \frac{2}{\sqrt{2+\sqrt{2}+\sqrt{2}}} \cdot \cdot$$

Do other numbers have similar expansions?