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MATH 350 : REAL ANALYSIS

Solution Set 9

(1) In class, we outlined a proof of the Cauchy criterion which is similar to that given in Chapter 19 of the book. The biggest difference between our proof and the book's is the approach to the Bolzano-Weierstrass theorem; our proof was based on Miles' and Ben's insight that any sequence has a monotone subsequence, while the book's proof is a clever binary search algorithm. The goal of this problem is to make our approach rigorous. Given a sequence (a_n), we call a_k a *peak* iff $a_k \geq a_m$ for all $m \geq k$.

(a) Suppose (a_n) has infinitely many peaks. Prove that (a_n) has a monotone subsequence.

By definition of peak, the subsequence consisting of the peaks forms a monotonically decreasing sequence.

(b) Suppose (a_n) has finitely many peaks. Prove that (a_n) has a monotone subsequence.

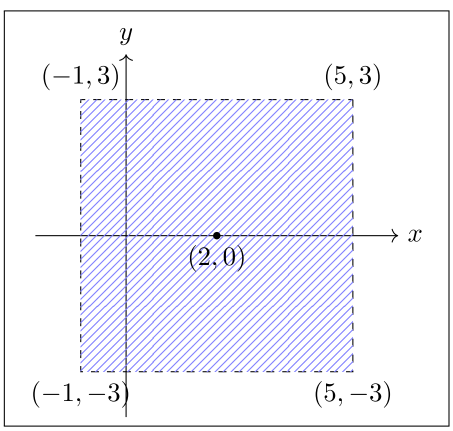
There exists some largest M such that a_M is a peak. Let $m_1 = M + 1$. Since a_{m_1} isn't a peak, there exists $m_2 > m_1$ such that $a_{m_1} < a_{m_2}$. Since a_{m_2} isn't a peak, there exists $m_3 > m_2$ such that $a_{m_2} < a_{m_3}$. Since there are no more peaks we can continue this process indefinitely, thus creating a monotonically increasing subsequence (a_{m_k}).

(c) Deduce the Bolzano-Weierstrass theorem from the previous parts.

Given a bounded sequence (a_n), our work above demonstrates that it has a monotone subsequence (a_{n_k}). Since (a_{n_k}) must be bounded, the Monotone Convergence Theorem implies it must converge. Thus we've found a convergent subsequence of (a_n).

(2) For each of the following metrics on $\mathbb{R}^2$, draw a picture the open ball $\mathcal{B}_3\big((2,0)\big)$. No proofs necessary.

(a) The chessboard metric $d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.

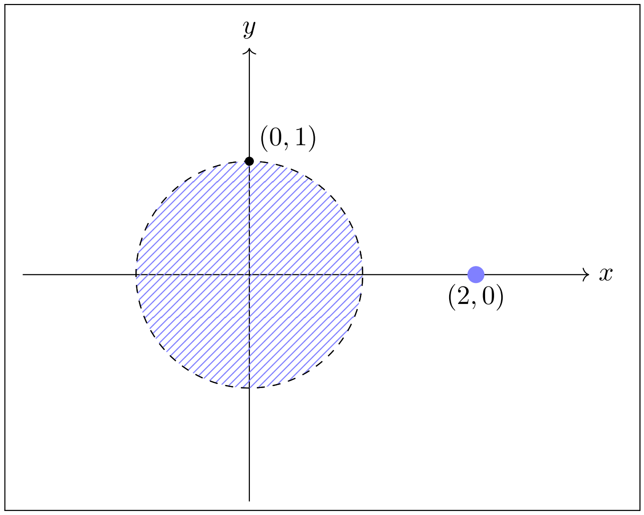


This open ball is the interior of a square of side length 6, centered at $(2, 0)$, not including any of the boundary.

(b) The British Rail metric

$$d(x, y) := \begin{cases} |x| + |y| & \text{if } x \neq y \\ 0 & \text{otherwise.} \end{cases}$$

(Here $|x|$ denotes the Euclidean distance from x to the origin.)



This open ball is the single point $(2, 0)$ union with the interior of the unit circle centered at the origin (not including any of the boundary).

(c) The discrete metric $d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$

By definition, $d(x, y) \leq 1 \; \forall x, y \in \mathbb{R}^2$. Thus, the open ball is all of $\mathbb{R}^2$.

(3) Suppose (X, d) is a metric space and $\mathcal{A} \subseteq X$. We say $p \in X$ is an *interior point* of $\mathcal{A}$ iff $\exists r > 0$ such that $\mathcal{B}_r(p) \subseteq \mathcal{A}$, and that $p \in X$ is a *limit point* of $\mathcal{A}$ iff there exists a sequence (a_n) of points in $\mathcal{A}$ such that $\lim_{n \to \infty} a_n = p$. (As always, $\mathcal{B}_r(p)$ denotes the ball of radius r around p.)

(a) Prove that $\mathcal{A}$ is open iff every point of $\mathcal{A}$ is an interior point of $\mathcal{A}$. (In class we defined: $\mathcal{A}$ is open iff $\partial\mathcal{A} \cap \mathcal{A} = \varnothing$.)

($\Rightarrow$) Suppose $\mathcal{A}$ is open. Pick any $a \in \mathcal{A}$. By hypothesis, $a \notin \partial\mathcal{A}$, so there is an $\epsilon > 0$ such that $\mathcal{B}_\epsilon(a) \subseteq \mathcal{A}$; in other words, a is an interior point of $\mathcal{A}$. Thus every point of $\mathcal{A}$ is an interior point.

($\Leftarrow$) By definition, any interior point of $\mathcal{A}$ has a ball of some radius $\epsilon > 0$ around it such that the ball is entirely contained in $\mathcal{A}$, which means an interior point cannot lie on $\partial\mathcal{A}$. Thus, if every point of $\mathcal{A}$ is an interior point, then no point of $\mathcal{A}$ is on the boundary of $\mathcal{A}$. In other words, $\mathcal{A} \cap \partial\mathcal{A} = \emptyset$, whence $\mathcal{A}$ is open.

(b) Prove that $\mathcal{A}$ is closed iff every limit point of $\mathcal{A}$ is in $\mathcal{A}$. (In class we defined: $\mathcal{A}$ is closed iff $\partial\mathcal{A} \subseteq \mathcal{A}$.)

($\Rightarrow$) Suppose $\mathcal{A}$ has a limit point p that is not in the set. Then for any $\epsilon > 0$, the ball of radius ϵ about p must contain a point in $\mathcal{A}$, by definition of p being a limit point. But that ball also contains p itself, which is in $\mathcal{A}^c$. Thus the ball intersects both $\mathcal{A}$ and $\mathcal{A}^c$, hence p is on the boundary of $\mathcal{A}$. Since $\mathcal{A}$ does not contain one of its boundary points, it is not closed. Thus any closed set must contain all its limit points.

($\Leftarrow$) Suppose $\mathcal{A}$ is not closed. Then there is some point $b \in \partial\mathcal{A} \setminus \mathcal{A}$. Since b is on the boundary of $\mathcal{A}$, for any $\epsilon > 0$, the ball of radius ϵ about b intersects $\mathcal{A}$. Let n_ϵ be a point in the intersection of the ball of radius ϵ and $\mathcal{A}$. Then the sequence $n_1, n_{1/2}, n_{1/3}, \ldots$ is a sequence in $\mathcal{A}$ converging to b, so b is a limit point of $\mathcal{A}$. Thus $\mathcal{A}$ does not contain all its limit points. Thus any set that contains all its limit points must be closed.

(4) Suppose (X, d) is a metric space. Prove that $\mathcal{B}_r(p)$ is open for any $p \in X$ and any $r > 0$.

Pick any $q \in \mathcal{B}_r(p)$, and set $\epsilon := r - d(p,q)$.

Claim. $\mathcal{B}_\epsilon(q) \subseteq \mathcal{B}_r(p)$

Proof. Pick $x \in \mathcal{B}_\epsilon(q)$. Then

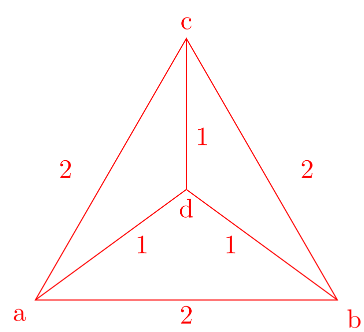
$$d(x,p) \leq d(x,q) + d(p,q) < \epsilon + d(p,q) = r. \qquad \square$$

Thus, we've shown that every point of $\mathcal{B}_r(p)$ is interior; it follows that $\mathcal{B}_r(p)$ is open.

(5) Decide (with proof or counterexample) whether each of the following is a metric space.

(a) The set $\{a, b, c, d\}$ with the distance between any two of a, b, c being 2 and the distance between d and any one of a, b, c being 1. (The distance between any element and itself is 0, of course.)

Call the distance function ϕ (just to avoid confusion with the point d). I claim $(\underbrace{\{a,b,c,d\}}_{=:X}, \phi)$ is a metric space. Let's look at a visual interpretation first.



Ok, now let's prove (X, ϕ) form a metric space. The first two properties are handed to us for free, so it remains to show triangle inequality. Here's a visual way to think about triangle inequality. Triangle inequality is violated if there are two points $p_1, p_2 \in X$ such that the fastest way to get from p_1 to p_2 is *not* by directly taking the path from p_1 to p_2. For example, let's choose $a, d \in X$. There are many ways to get from a to d. You can go from $a \to c \to b \to d$, but that takes distance $2+2+1$, whereas it's faster to go just from $a \to d$ (takes distance 1). By inspection, there are no two points that break triangle inequality, hence (X, ϕ) is a metric space.

(b) $\mathbb{R}^\infty := \{(a_n) : (a_n) \text{ is a sequence of real numbers}\}$, with respect to $d(x,y) := \max\{|x_n - y_n|\}$.

This isn't well-defined: the sequences $x_n = 0$ and $y_n = n$ would get infinitely far apart under this metric, but the codomain of any metric must be $\mathbb{R}$ and $\infty \notin \mathbb{R}$!

(c) $\mathcal{F} := \{A \subseteq \mathbb{Z} : A \text{ is finite and nonempty}\}$, with respect to $d(X,Y) := \log \frac{|X-Y|}{\sqrt{|X|}\sqrt{|Y|}}$. Here $|S|$ denotes the size of S and $X - Y := \{x - y : x \in X, y \in Y\}$.

No, this is not a metric, because $d(A,A)$ might be nonzero. For example, let $A := \{2,3\}$. Then $A - A = \{0, \pm 1\}$, so $d(A,A) = \log \frac{3}{2} \neq 0$.

Notes. Remarkably, this function (called the *Ruzsa distance*) satisfies all the other properties, including the triangle inequality.

(6) Exploring metrics on $\mathbb{R}^2$.

(a) Prove that the Euclidean metric on $\mathbb{R}^2$ is, in fact, a metric.

By definition,

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

This equals 0 iff both $(x_1 - y_1)^2, (x_2 - y_2)^2 = 0$, which happens iff $x_1 = y_1$ and $x_2 = y_2$. It's also clear that this metric is symmetric: $d(x,y) = d(y,x)$. It remains to prove the triangle inequality.

The most direct approach is quite algebraically involved. However, we can simplify this significantly by observing that the Euclidean distance is translation invariant, i.e. that $d(x,y) = d(x-z, y-z)$ for any z. After translating appropriately, we see that triangle inequality is equivalent to showing that

$$d(x,y) \leq d(x,0) + d(0,y) \tag{$*$}$$

for all x, y.

Note that $(x_1 y_2 - x_2 y_1)^2 \geq 0$. It follows that

$$(x_1 y_1 + x_2 y_2)^2 \leq (x_1^2 + x_2^2)(y_1^2 + y_2^2). \tag{$\dagger$}$$

In particular,

$$-x_1 y_1 - x_2 y_2 \leq \sqrt{(x_1^2 + x_2^2)(y_1^2 + y_2^2)}.$$

From here, it's straightforward to deduce

$$\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \leq \sqrt{x_1^2 + x_2^2} + \sqrt{y_1^2 + y_2^2}$$

which is precisely $(*)$.

Notes. One nice interpretation of $(\dagger)$ is in the language of linear algebra:

$$\vec{x} \cdot \vec{y} \leq |\vec{x}| \cdot |\vec{y}|$$

where the left hand side is the dot product, while the right hand side is ordinary multiplication on $\mathbb{R}$.

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NOTES. Above, we saw that (†) implies (∗); it turns out that the converse implication holds as well, so the inequality (†) is equivalent to the triangle inequality for the Euclidean metric on $\mathbb{R}^2$. Similarly, it turns out the triangle inequality for the Euclidean metric on $\mathbb{R}^n$ is equivalent to the following:

Lemma (Cauchy-Schwarz inequality)**.** *For any real numbers a_i, b_i, we have*

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

(or equivalently, $\vec{a} \cdot \vec{b} \leq |\vec{a}| \cdot |\vec{b}|$).

(b) Suppose $\mathcal{O}$ is a subset of $\mathbb{R}^2$ that's open with respect to the Euclidean metric. Must it also be open with respect to the taxicab metric?

Suppose $\mathcal{O} \subseteq \mathbb{R}^2$ is open with respect to the Euclidean metric. Pick any $\alpha \in \mathcal{O}$; we claim that α is in the interior of $\mathcal{O}$ with respect to the taxicab metric. Since $\mathcal{O}$ is open with respect to the Euclidean metric, $\exists \delta > 0$ such that the open euclidean ball of radius δ around α is entirely contained inside $\mathcal{O}$.

Consider the open taxicab ball of radius δ around α. Pick any x in this ball; by definition, the taxicab distance between x and α is smaller than δ, i.e.

$$|x_1 - \alpha_1| + |x_2 - \alpha_2| < \delta.$$

Squaring both sides, we deduce

$$|x_1 - \alpha_1|^2 + |x_2 - \alpha_2|^2 \leq |x_1 - \alpha_1|^2 + |x_2 - \alpha_2|^2 + 2|x_1 - \alpha_1| \cdot |x_2 - \alpha_2| < \delta^2.$$

This implies that x lies in the Euclidean ball of radius δ around α, which we know is entirely contained in $\mathcal{O}$. We've therefore shown that every point in the taxicab ball of radius δ around α is contained entirely in $\mathcal{O}$; it follows that α is an interior point of $\mathcal{O}$ with respect to the taxicab metric, as desired.

NOTES. All this becomes much more clear when looking at pictures: the taxicab open ball is the largest diamond that fits inside the Euclidean ball of the same radius.

(c) The Euclidean and taxicab metrics on $\mathbb{R}^2$ both have the form

$$d_p(x, y) := \left(|x_1 - y_1|^p + |x_2 - y_2|^p\right)^{1/p}$$

(d_1 is the taxicab metric, d_2 is the Euclidean metric). It turns out that d_p is a metric for every real number $p \geq 1$. (Don't worry about proving it here, although it is a fun challenge to think about when you have some spare time.) Can you describe any of the other metrics on $\mathbb{R}^2$ that we've encountered (chessboard, British Rail, and discrete) in terms of d_p? No formal proofs necessary, but give a bit of justification for your answer.

This is open ended, of course, but the cleanest answers are those that describe the metric in terms of d_p *without* reference to specific inputs.

Claim. The **chessboard metric** is $d_\infty := \lim_{p\to\infty} d_p$.

Proof. If $x = y$ then $d_p(x, y) = 0$ for all p, so the limit is also 0. If $x \neq y$, then without loss of generality we have

$$\max\{|x_1 - y_1|, |x_2 - y_2|\} = |x_1 - y_1| > 0.$$

Then

$$d_p(x, y) = |x_1 - y_1| \left(1 + \left(\frac{|x_2 - y_2|}{|x_1 - y_1|}\right)^p\right)^{1/p}.$$

Since this is bounded above by $|x_1 - y_1| \cdot 2^{1/p}$ and bounded below by $|x_1 - y_1|$, we see $d_p(x, y) \to |x_1 - y_1|$ as $p \to \infty$. □

We can also express the **discrete metric** in terms of d_p (albeit in a more artificial form) as $\min\{\lceil d_2 \rceil, 1\}$.

NOTES.The metric d_p is called the ℓ^p metric; you will explore it in virtually any advanced course on analysis.

(7) Given a metric space (X, d) where X has at least 3 elements. Prove that there exists a metric on X that's not a scalar multiple of d or of the discrete metric.

Observe that rescaling a metric doesn't affect its metric properties. Also, it's easy to see that summing any two metrics produces a metric. Thus, any linear combination $D(x, y) := \alpha d_1(x, y) + \beta d_2(x, y)$ of any two metrics d_1, d_2 is a metric as well. In particular, if the given metric d isn't the discrete metric, then the sum of d and the discrete metric produces a new metric on X.

However, if d *is* the discrete metric, then we haven't solved the problem, since in this case the sum of d and the discrete metric would be a scalar multiple of d! So in this case, we have to do something more clever. There are many approaches to this; here's one.

Given a metric d, set $D(x, y) := \frac{d(x,y)}{1+d(x,y)}$. I claim that D is a metric. It's easy to verify the first two properties, so it suffices to handle the triangle inequality:

$$\begin{aligned} D(x, z) = 1 - \frac{1}{1 + d(x, z)} &\leq 1 - \frac{1}{1 + d(x, y) + d(y, z)} = \frac{d(x, y) + d(y, z)}{1 + d(x, y) + d(y, z)} \\ &= \frac{d(x, y)}{1 + d(x, y) + d(y, z)} + \frac{d(y, z)}{1 + d(x, y) + d(y, z)} \\ &\leq \frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, z)}{1 + d(y, z)} = D(x, y) + D(y, z). \end{aligned}$$

It's an exercise to check that $D \neq d$ and also cannot equal the discrete metric.

(8) Given (X, d) a metric space and $\mathcal{A} \subseteq X$. Prove that $\mathcal{A}$ is closed iff $\mathcal{A}^c$ is open.

We warm up with the following useful observation:

Lemma. *For any set $\mathcal{A} \subseteq X$, we have $\partial\mathcal{A} = \partial\mathcal{A}^c$.*

Proof. Let $x \in \partial\mathcal{A}$. Then by the definition of a boundary point of $\mathcal{A}$, for any $\epsilon > 0$, we have $B_\epsilon(x) \cap \mathcal{A} \neq \emptyset$ and $B_\epsilon(x) \cap \mathcal{A}^c \neq \emptyset$. Since $(\mathcal{A}^c)^c = \mathcal{A}$, we can just as well write that $B_\epsilon(x) \cap \mathcal{A}^c \neq \emptyset$ and $B_\epsilon(x) \cap (\mathcal{A}^c)^c \neq \emptyset$. But this is exactly the definition for x to be in the boundary of $\mathcal{A}^c$. Thus $\partial\mathcal{A} \subseteq \partial\mathcal{A}^c$. Replacing $\mathcal{A}$ with $\mathcal{A}^c$, we have that $\partial\mathcal{A}^c \subseteq \partial(\mathcal{A}^c)^c$. Again, $(\mathcal{A}^c)^c = \mathcal{A}$, so that means $\partial\mathcal{A}^c \subseteq \partial\mathcal{A}$. We now have subsets in both directions, so we conclude $\partial\mathcal{A} = \partial\mathcal{A}^c$. $\square$

Now we turn to the given problem. Suppose $\mathcal{A}$ is open. Then $\partial\mathcal{A} \cap A = \emptyset$ by definition of being an open set. That means all of $\partial\mathcal{A}$ is in $\mathcal{A}^c$. Since $\partial\mathcal{A} = \partial\mathcal{A}^c$ by our lemma, we have that $\mathcal{A}^c$ contains all of its own boundary, hence it is closed. Now suppose $\mathcal{A}$ is closed. Then it contains all of its boundary, so $\mathcal{A}^c$ contains none of the shared boundary, hence $\mathcal{A}^c \cap \partial\mathcal{A}^c = \emptyset$ and $\mathcal{A}^c$ is open.

NOTES. Note that most sets that you encounter in the wild are *neither* open nor closed!

(*) (*Optional challenge problem—won't be graded*) Let $M_{n\times n}$ denote the space of all $n \times n$ matrices with real entries. Prove that $d(x, y) := \operatorname{rank}(x - y)$ is a metric on $M_{n\times n}$.

Left to the interested reader to tackle over winter break.