$\mathbb{R}$ IS THE ONLY COMPLETE ORDERED FIELD

LEO GOLDMAKHER

ABSTRACT. We prove that any two complete ordered fields are isomorphic to one another. Put differently: $\mathbb{R}$ is the only complete ordered field (up to isomorphism).

1. INTRODUCTION

In a typical introductory real analysis course, one defines $\mathbb{R}$ to be a complete ordered field. This seems like a very fishy definition; it's conceivable that there exist many different complete ordered fields, so which one should we take to be $\mathbb{R}$? It turns out that it doesn't matter which one you choose, because they're all *isomorphic*: any two complete ordered fields are indistinguishable apart from the names assigned to the elements and the operations. One way to think about this is that any complete ordered field is $\mathbb{R}$ described in some language; maybe one is described in English and another in Russian, but they're both still $\mathbb{R}$. Here's a more precise version of this statement:

Theorem 1.1. *If $\mathbb{R}$ and k are complete ordered fields, then there exists a bijection $\varphi : \mathbb{R} \hookrightarrow\!\!\!\!\rightarrow k$ such that*

(i) $x < y$ *implies* $\varphi(x) < \varphi(y)$
(ii) $\varphi(x + y) = \varphi(x) + \varphi(y)$
(iii) $\varphi(xy) = \varphi(x)\varphi(y)$.

In terms of the previous analogy, you can think of $\mathbb{R}$ as the real numbers described in English and k as the real numbers described in Russian. The function φ serves as an English-Russian dictionary that translates from English to Russian; the fact that it's a bijection means that each number in English corresponds to precisely one number in Russian and vice-versa. The three properties of φ guarantee that not only do all the individual numbers match up between the two languages, but the relationships among the numbers are independent of the language used to describe them.

2. WARM-UP: THE RATIONAL NUMBERS

We assume the existence of $\mathbb{R}$ and its famous subsets $\mathbb{Q}$ (the rationals), $\mathbb{Z}$ (the integers), and $\mathbb{Z}^+$ (the positive integers). Suppose k is a complete ordered field. Then there exists an additive identity 0_k and a multiplicative identity 1_k. For any $n \in \mathbb{Z}^+$, adding 1_k to itself n times produces a number

$$n_k := \underbrace{1_k + 1_k + \cdots + 1_k}_{n} \in k.$$

The collection of all such elements of k is an analogue of the positive integers inside k:

$$\mathbb{Z}_k^+ := \{n_k : n \in \mathbb{Z}^+\}.$$

By adjoining the additive inverses of all these elements along with the additive identity, we obtain the analogue of the integers:

$$\mathbb{Z}_k := \mathbb{Z}_k^+ \cup \{0_k\} \cup \{-m : m \in \mathbb{Z}_k^+\}.$$

From now on, I'll refer to any element of $\mathbb{Z}_k$ as a *k-integer*. Finally, taking all possible ratios between k-integers we obtain the k-rationals:

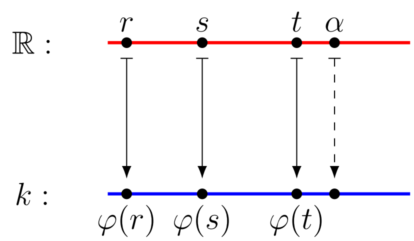
$$\mathbb{Q}_k := \left\{\frac{a}{b} : a \in \mathbb{Z}_k, b \in \mathbb{Z}_k^+\right\}.$$

Note that we have a natural bijection $\mathbb{Q} \hookrightarrow\!\!\!\!\rightarrow \mathbb{Q}_k$, with positive integers n corresponding to n_k, negative integers $-n$ corresponding to $-n_k$, 0 corresponding to 0_k, and ratios of two integers corresponding to the ratio of the two corresponding k-integers. Moreover, this correspondence satisfies all three conditions in Theorem 1.1, for a superficial reason: for any proof we write down concerning elements of $\mathbb{Q}$, we could simply replace every integer that occurs in the proof by its corresponding k-integer and the proof would yield the analogous result for $\mathbb{Q}_k$. Thus, for example, the proof that $\frac{2}{3} > \frac{1}{2}$ also yields $\frac{2_k}{3_k} > \frac{1_k}{2_k}$. (If this isn't obvious to you, I encourage you to write down a proof in $\mathbb{Q}$, and then ask yourself: which step of the proof would change if you replaced all integers by the corresponding k-integers?)

Indeed, *any* result concerning $\mathbb{Q}$ and its relation to $\mathbb{R}$ must continue to hold for $\mathbb{Q}_k$ and its relation to k, because the proofs rely only on the axioms *satisfied* by $\mathbb{Q}$ and $\mathbb{R}$, not on what these sets actually *are*. Thus, for example, $\mathbb{Q}_k$ must be dense in k (i.e. there must exist a k-rational strictly between any two elements of k), and the Archimedean property holds (i.e. there exist arbitrarily large k-integers and arbitrarily tiny k-rationals).

3. Extending the map to all of $\mathbb{R}$

OK, so now we have a nice bijection $\mathbb{Q} \hookrightarrow\!\!\!\!\rightarrow \mathbb{Q}_k$. But what we're actually after is a bijection between supersets of these: we want some $\varphi : \mathbb{R} \hookrightarrow\!\!\!\!\rightarrow k$. How might we construct one? Here's a picture:



From the picture, we know precisely where α should be sent: it should go to the spot just above where all the numbers smaller than α get sent, i.e., $\varphi(\alpha) = \sup\{\varphi(q) : q < \alpha\}$. This looks like a pretty solid definition at first glance, but there's a major flaw: we don't know what $\varphi(q)$ is for most numbers $q < \alpha$, since we're currently trying to *define* where to send an arbitrary real number! Fortunately, there is one type of number for which we understand very well where it gets sent – rational numbers. This motivates the following definition:

$$\varphi(\alpha) := \sup\{\varphi(q) : q < \alpha, q \in \mathbb{Q}\}.$$

We're almost ready to prove Theorem 1.1, but there's one fundamental thing to check first: that the supremum exists. For ease of reference, let's label the set in question: for a given $\alpha \in \mathbb{R}$, let

$$S(\alpha) := \{\varphi(q) : q < \alpha, q \in \mathbb{Q}\}.$$

Thus, $\varphi(\alpha) := \sup S(\alpha)$.

Claim. $\sup S(\alpha) \in k$; in other words, φ is well-defined.

Proof. By the completeness axiom, it suffices to prove that for any given $\alpha \in \mathbb{R}$, $S(\alpha)$ is non-empty and bounded above.

• *$S(\alpha)$ is non-empty.* By the Archimedean property there exists some positive integer $n > |\alpha|$. It follows that $-n < \alpha$, whence $-n_k \in S(\alpha)$.

• *$S(\alpha)$ is bounded above.* By the Archimedean property there exists some positive integer $n > \alpha$. It follows that $n_k > \varphi(q)$ for all $q \in \mathbb{Q} \cap (-\infty, \alpha)$, whence n_k is an upper bound on $S(\alpha)$. □

4. Proof of Theorem 1.1

We've now constructed a well-defined function $\varphi : \mathbb{R} \to k$; our goal is to verify that it satisfies the conclusions of Theorem 1.1. As we proceed, keep in mind that the function φ is a bijection between the rationals and the k-rationals, and moreover satisfies the conclusions of Theorem 1.1 for all rational inputs.

The first natural claim to attack is that φ is a bijection. Playing around with injectivity, one quickly sees that φ is injective because it is order-preserving: if $\alpha < \beta$ then $\varphi(\alpha) < \varphi(\beta)$. Thus, we start there instead.

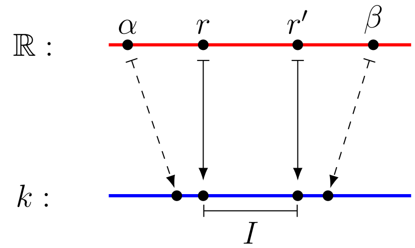
Lemma 4.1. *If $\alpha < \beta$ then $\varphi(\alpha) < \varphi(\beta)$.*

Before giving a rigorous proof, we work through the idea. Observe that

$$\alpha < \beta \implies S(\alpha) \subset S(\beta) \implies \sup S(\alpha) \le \sup S(\beta) \implies \varphi(\alpha) \le \varphi(\beta). \tag{4.1}$$

This is pretty good, but isn't quite as strong as we'd like because it leaves open the possibility that $\alpha < \beta$ but $\varphi(\alpha) = \varphi(\beta)$. So we have to work a bit harder.

How do we prove that φ is strictly order-preserving? Well, we already know that this holds for rational inputs. The given α, β might not be rational, but between them there are many rationals, and the order of these will be preserved. This yields the main idea of our proof: we'll produce two rational numbers r and r' between α and β, and since the images of these two rationals have some space between them, that forces there to be some space between $\varphi(\alpha)$ and $\varphi(\beta)$ as well. In the picture below, the interval I between the images of r and r' *cannot* contain the images of α and β.



We're ready for a rigorous proof.

Proof. Given real numbers $\alpha < \beta$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $r \in (\alpha, \beta) \cap \mathbb{Q}$ and $r' \in (r, \beta) \cap \mathbb{Q}$. From (4.1) we deduce

$$\varphi(\alpha) \le \varphi(r) \qquad \text{and} \qquad \varphi(r') \le \varphi(\beta).$$

Moreover, since φ is order-preserving on $\mathbb{Q}$, we have $\varphi(r) < \varphi(r')$. It follows that $\varphi(\alpha) < \varphi(\beta)$. ☐

Armed with this tool, it becomes relatively straightforward to prove that φ is a bijection.

Proposition 4.2. *φ is a bijection.*

Remark. For ease of reference, I'll use the letters q, r, s to denote rationals, α, β, γ to denote arbitrary real numbers, and Fraktur font ($\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{q}$, $\mathfrak{r}$ etc.) to denote elements of k.

Proof. Since φ is order-preserving, it's clearly injective. Thus it remains to prove that φ is surjective. Pick $\mathfrak{a} \in k$; we wish to produce an $\alpha \in \mathbb{R}$ such that $\varphi(\alpha) = \mathfrak{a}$. Consider the set $\mathbb{Q}_k \cap (-\infty, \mathfrak{a})$ (i.e. all k-rationals smaller than $\mathfrak{a}$) and its preimage:

$$\varphi^{-1}\Big(\mathbb{Q}_k \cap (-\infty, \mathfrak{a})\Big) := \Big\{\varphi^{-1}(\mathfrak{q}) : \mathfrak{q} \in \mathbb{Q}_k \cap (-\infty, \mathfrak{a})\Big\} \subseteq \mathbb{Q}.$$

Now let α be the supremum of this set, i.e.

$$\alpha := \sup\left(\varphi^{-1}\Big(\mathbb{Q}_k \cap (-\infty, \mathfrak{a})\Big)\right).$$

It is a good exercise to show that $\alpha \in \mathbb{R}$ and that $\varphi(\alpha) = \mathfrak{a}$. ☐

It remains to prove that φ plays nice with addition and multiplication. We handle addition first.

Proposition 4.3. *$\varphi(\alpha + \beta) = \varphi(\alpha) + \varphi(\beta)$ for all $\alpha, \beta \in \mathbb{R}$.*

Proof. Once again we rely on the fact that φ satisfies all the properties we desire at rational inputs. In particular,

$$\varphi(q + q') = \varphi(q) + \varphi(q') \qquad \forall q, q' \in \mathbb{Q};$$

our task is to show this continues to hold for arbitrary real inputs. Recall that $\varphi(x) := \sup S(x)$, where $S(x) := \{\varphi(q) : q < x, q \in \mathbb{Q}\}$. Thus we're trying to prove

$$\sup S(\alpha + \beta) = \sup S(\alpha) + \sup S(\beta)$$

holds for arbitrary $\alpha, \beta \in \mathbb{R}$. I claim (see below) that $S(\alpha + \beta) = S(\alpha) + S(\beta)$; the claim immediately follows from this, since $\sup(X + Y) = \sup X + \sup Y$ for any two sets X and Y. ☐

Lemma 4.4. *$S(\alpha+\beta) = S(\alpha) + S(\beta)$ for all $\alpha, \beta \in \mathbb{R}$.*

Proof. First, observe that if $x \in S(\alpha) + S(\beta)$, then $x = \varphi(q_1) + \varphi(q_2)$ for some rational numbers $q_1 < \alpha$ and $q_2 < \beta$. It follows that $q_1 + q_2 < \alpha + \beta$, whence

$$x = \varphi(q_1) + \varphi(q_2) = \varphi(q_1 + q_2) \in S(\alpha + \beta),$$

verifying that $S(\alpha) + S(\beta) \subseteq S(\alpha + \beta)$.

Next, we verify the reverse inclusion. Suppose $y \in S(\alpha+\beta)$, so that $y = \varphi(q)$ for some rational $q < \alpha + \beta$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a rational number $q' \in (q - \alpha, \beta)$. Note that $q' < \beta$ and $q - q' < \alpha$, whence $\varphi(q - q') \in S(\alpha)$ and $\varphi(q') \in S(\beta)$. It follows that

$$y = \varphi(q) = \varphi(q') + \varphi(q - q') \in S(\alpha) + S(\beta).$$

Thus, $S(\alpha + \beta) \subseteq S(\alpha) + S(\beta)$. This concludes the proof. □

We finish by proving

Proposition 4.5. *$\varphi(\alpha\beta) = \varphi(\alpha)\varphi(\beta)$ for all $\alpha, \beta \in \mathbb{R}$.*

Remark. Note that this argument cannot follow the model of the previous one, because $\sup XY \neq (\sup X)(\sup Y)$ in general. Instead, we proceed more directly. Recall that $\varphi(\alpha\beta) = \sup S(\alpha\beta)$ by definition. We'll prove that $\varphi(\alpha)\varphi(\beta)$ is also the supremum of $S(\alpha\beta)$. As usual, we'll do this in two steps: first, we'll prove that $\varphi(\alpha)\varphi(\beta)$ is an upper bound on $S(\alpha\beta)$, and then we'll prove that anything smaller cannot be an upper bound.

Lemma 4.6. *$\varphi(q) < \varphi(\alpha)\varphi(\beta)$ for any $\alpha, \beta \in \mathbb{R}$ and any rational number $q < \alpha\beta$. In other words, $\varphi(\alpha)\varphi(\beta)$ is an upper bound on $S(\alpha\beta)$.*

Proof. We prove the claim for the special case $\alpha, \beta > 0$, and leave the rest as an exercise to the reader.

Pick any rational $q < \alpha\beta$, and then another rational $q_1 \in \left(\frac{q}{\beta}, \alpha\right)$. Then $q_2 := \frac{q}{q_1} < \beta$. In other words, given a rational $q < \alpha\beta$, we can express it in the form $q = q_1 q_2$ with some rational numbers $q_1 < \alpha$ and $q_2 < \beta$. It follows that

$$\varphi(\alpha)\varphi(\beta) > \varphi(q_1)\varphi(q_2) = \varphi(q_1 q_2) = \varphi(q).$$ □

Lemma 4.7. *For any $\epsilon_k > 0_k$, there exists a rational $q < \alpha\beta$ such that $\varphi(q) > \varphi(\alpha)\varphi(\beta) - \epsilon_k$. In other words, anything smaller than $\varphi(\alpha)\varphi(\beta)$ isn't an upper bound on $S(\alpha\beta)$.*

Proof. We prove the lemma for the special case $\alpha, \beta > 0$, and leave the rest as an exercise to the reader.

Given $\epsilon_k > 0_k$. Set $\epsilon := \varphi^{-1}(\epsilon_k) \in \mathbb{R}$, and note that $\epsilon > 0$. Now pick any positive rational number

$$\delta < \min\left\{\frac{\epsilon}{\alpha + \beta + 1}, 1\right\},$$

as well as rational numbers $q_1 \in (\alpha - \delta, \alpha)$ and $q_2 \in (\beta - \delta, \beta)$. Then $\alpha < q_1 + \delta$ and $\beta < q_2 + \delta$, whence

$$\varphi(\alpha)\varphi(\beta) < \varphi(q_1 + \delta)\varphi(q_2 + \delta).$$

The key observation is that $q_1 + \delta$ and $q_2 + \delta$ are both rational, and we already know that φ preserves multiplication on rational inputs. Thus,

$$\begin{aligned}\varphi(\alpha)\varphi(\beta) &< \varphi(q_1 + \delta)\varphi(q_2 + \delta) \\ &= \varphi(q_1 q_2) + \varphi\big((q_1 + q_2)\delta + \delta^2\big) \\ &< \varphi(q_1 q_2) + \varphi\big((\alpha + \beta)\delta + \delta\big) \\ &= \varphi(q_1 q_2) + \varphi\big((\alpha + \beta + 1)\delta\big) \\ &< \varphi(q_1 q_2) + \varphi(\epsilon) \\ &= \varphi(q_1 q_2) + \epsilon_k\end{aligned}$$ □

Combining Lemmas 4.6 and 4.7, we deduce that

$$\varphi(\alpha)\varphi(\beta) = \sup S(\alpha\beta).$$

But the right hand side is $\varphi(\alpha\beta)$, by definition. This concludes the proof of Proposition 4.5. Putting all our work together yields Theorem 1.1. QEFD

DEPT OF MATHEMATICS AND STATISTICS, WILLIAMS COLLEGE, WILLIAMSTOWN, MA, USA 01267
Email address: Leo.Goldmakher@williams.edu