A TOPOLOGICAL PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA

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ABSTRACT. The goal of this note is to write a brief sketch of the topological proof of the Fundamental Theorem of Algebra. Details will be kept to a minimum.

1. THE INTUITION

Recall that the Fundamental Theorem of Algebra asserts that $\mathbb{C}$ is algebraically closed, or in other words, that any non-constant polynomial with complex coefficients can be factored as a product of linear polynomials with complex coefficients. At the start of the semester we explored a hands on explanation of why this is true. The idea was as follows. Let C_r denote the circle of radius r in the complex plane centered at 0, and suppose $p \in \mathbb{C}[x]$ has degree n and doesn't vanish anywhere in $\mathbb{C}$. Then $p(C_0)$ consists of the single point $p(0) \in \mathbb{C} \setminus \{0\}$, while $p(C_r)$ looks more and more like C_{r^n} as $r \to \infty$. Thus, choosing a huge value R and letting r continuously vary from R to 0, the images $p(C_r)$ continuously deforms from a huge loop winding around 0 to the single non-zero point $p(0)$. It's intuitively clear that somewhere during the course of this deformation, there will be an s with $p(C_s)$ passing through 0. But this means that p would have a root! To see this argument visually, play around with this desmos example.

While this gives an excellent intuitive explanation for the fundamental theorem of algebra, it doesn't constitute a proof. How do we know that the loop must pass through 0 at some point during the course of its continuous deformation? More importantly, what does continuous even mean in this context? With the tools we've developed over the course of the semester, we can transform our intuitive argument into a rigorous one.

2. A RIGOROUS PROOF

The intuitive argument above should remind you very strongly of the idea of homotopy: as r goes from 0 to some fixed $R > 0$, we have a family of loops $p(C_r)$ that is continuously deforming. This isn't quite a homotopy, however. For one thing, $p(C_r)$ certainly looks like a loop, but it's a set rather than a function. This is relatively straightforward to fix: parametrizing a circle is easy, and we can turn this into a parametrization of $p(C_r)$. A second issue is that the family of loops $p(C_r)$ don't share a base point, which was one of our requirements for homotopies. This isn't so bad, either: we can renormalize each loop so that it lives in C_1, and then we can rig up a common basepoint. With this preparation, we're ready to walk through the rigorous proof!

As before, suppose $p \in \mathbb{C}[x]$ is a polynomial that has no roots in $\mathbb{C}$; our goal is to deduce that p must be constant. To simplify notation, we further assume p is monic, i.e. that its highest-degree coefficient is 1. (Note that to prove that FTA in general, it suffices to prove it for monic polynomials.) Since the projection $\pi : \mathbb{C} \setminus \{0\} \to C_1$ defined by $x \mapsto \frac{x}{|x|}$ is continuous, and by hypothesis $p : \mathbb{C} \to \mathbb{C} \setminus \{0\}$, we deduce that $\pi \circ p : \mathbb{C} \to C_1$ is continuous. In other words, the function $\pi \circ p$ is a renormalization of p that allows us to model the behavior of p within the context of the unit circle C_1.

Next, we turn each set $p(C_r)$ into a loop. We start by parametrizing the circle C_r: define $\gamma_r : [0, 1] \to \mathbb{C}$ by

$$\gamma_r(t) := re^{2\pi i t}.$$

In words, γ_r is the loop that starts at the point r and winds around C_r counterclockwise precisely once, travelling at a constant rate the entire time. Immediately from this, we see that $p \circ \gamma_r$ is a loop that walks along the curve $p(C_r)$, starting and ending at the point $p(r)$. We deduce that $\pi \circ p \circ \gamma_r$ is a loop in C_1 based at the point $\frac{p(r)}{|p(r)|}$. Composing with the rotation $d_r : C_1 \to C_1$ defined $d_r(z) := \frac{z}{p(r)/|p(r)|}$, we've proved:

Proposition 2.1. *Let $H_r : [0, 1] \to C_1$ be defined $H_r := d_r \circ \pi \circ p \circ \gamma_r$. Then H_r is a loop in C_1 based at 1.*

In particular, H_0 is the trivial (constant) loop on C_1. Since H is a homotopy between H_0 and H_R for any particular fixed $R > 0$, we deduce

Corollary 2.2. *For every $r > 0$, the loop H_r in C_1 is homotopic to the trivial loop.*

The Fundamental Theorem of Algebra now follows almost immediately from the following

Lemma 2.3. *There exists $r > 0$ such that H_r is homotopic to $\underbrace{\gamma_1 * \gamma_1 * \cdots * \gamma_1}_{n \text{ times}}$, where $n = \deg p$ and $*$ is the concatenation operation on loops.*

We'll prove this lemma below, but first, we use it to give a short proof of the Fundamental Theorem of Algebra.

Proof of the Fundamental Theorem of Algebra. Combining Corollary 2.2 and Lemma 2.3, we deduce that

$$[\gamma_1]^n = [e] \tag{$\clubsuit$}$$

where e denotes the constant map. Under the isomorphism $\pi_1(C_1) \xrightarrow{\sim} \mathbb{Z}$ we have $[\gamma_1]^n \mapsto n$ and $[e] \mapsto 0$, so ($\clubsuit$) implies $n = 0$. We conclude that if $p \in \mathbb{C}[x]$ has no roots in $\mathbb{C}$, then p is constant. $\square$

Proof of Lemma 2.3. Write

$$p(z) = z^n + \underbrace{a_1 z^{n-1} + \cdots + a_{n-1} z + a_n}_{q(z)}.$$

By definition, we find that

$$H_r(t) = \frac{|r^n + q(r)|}{r^n + q(r)} \cdot \frac{r^n e^{2\pi i n t} + q(re^{2\pi i t})}{|r^n e^{2\pi i n t} + q(re^{2\pi i t})|}.$$

Note that if we erased all instances of q in this expression, it would simplify to

$$\underbrace{\gamma_1 * \gamma_1 * \cdots * \gamma_1}_{n \text{ times}}(t) = e^{2\pi i n t}.$$

This immediately tells us how to rig up a homotopy between these two maps: set

$$\varphi_m(t) := \frac{|r^n + mq(r)|}{r^n + mq(r)} \cdot \frac{r^n e^{2\pi i n t} + mq(re^{2\pi i t})}{|r^n e^{2\pi i n t} + mq(re^{2\pi i t})|}.$$

Then $\varphi_1 = H_r$ and $\varphi_0 = \gamma_1 * \gamma_1 * \cdots * \gamma_1$, and it's straightforward to check that $\varphi_m(0) = 1 = \varphi_m(1)$ for all m. We've found our homotopy!

Actually, there's a potential issue in our definition of φ_m: its denominator might vanish. When r is large, though, this can't happen, since for any $m \in [0, 1]$ and any $t \in \mathbb{R}$ we have

$$r^n e^{2\pi i n t} + mq(re^{2\pi i t}) = r^n e^{2\pi i n t} + O(r^{n-1}) \neq 0.$$

This concludes the proof of the Lemma. $\square$

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