Topology Notes 9/5/2024

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Summary

A couple technical things were said about the class, including that Leo's office has moved, and the class will cover point set and algebraic topology. We solved two problems together which involved continuous deformation as a key part of the solution, then moved onto a harder problem - the fundamental theorem of algebra - studying polynomials in \mathbb{C} as continuous deformations of complex circles. We made a "game plan" for a proof, but have yet to do it rigorously.

1 Technical details

1.1 Announcements!

- Leo's office has moved and is now Wachenheim 337.
- Friday at 12:30, senior math majors (exclusively) should grab free pizza in the science quad, where a presentation will be held about the colloquiums.
- Monday at 1:00, Satyan Devadoss from USD will give an accessible talk at Wachenheim 113: "Mysteries of Unfolding Polyhedra" open to everyone "math-interested". The problem is 500 years old and was famously explored by the Renaissance man Albrecht Dürer. This talk considers the question in higher dimensions.
- Saturday at 7:00PM, board games will take place at Paresky!
- A math book club will be held regularly after Purple Key Fair, time TBD! We will vote on books and then discuss them in chalk with snacks. Contact Isa by email to join the group chat.

1.2 Class setup

Topology 374 will have **weekly problem sets**, a **mid-term**, and a final. There will be no GLOW website for this course. There will be pre-set meetings with a TA (time TBD). Leo will have two office hours, one in Goodrich in the evening

and one traditional one in the day.

3/4 of the class will be on point-set topology, which studies the rigorous set-theoretic definitions of continuity and continuous deformations, but is no longer studied in pure topology. The other 1/4 will be spent on algebraic topology, which is constantly studied, particularly in its branches homotopy theory and cohomology theory.

2 Warm-up Questions: Cake and Strings

2.1 Imagine a cake. Into how many pieces can you cut it with...

1. One straight cut??

Hailey: Two pieces, like normal.Daniel: One, if you miss.Isaac: Since we're in topology, how might we deform the cake?Jenna: If the cake has bumps, you can cut it into as many as you want.



Figure 1: How to cut cake into 2, 1, and many pieces.

2. Two?

As per Isaac's idea, we'll assume the cake is a cylinder to make the problem more interesting. Although many people thought that the cake could only be cut into seven pieces, as would be true on a circle (shown left), it is in fact **possible to cut it into eight** (shown middle right). This is misleading; if the question were posed on a **sphere** (shown rightmost), which is a **continuous deformation of a cylinder**, the problem would seem trivial.



Figure 2: *a.* A circle can only be cut into 7 with 3 cuts. *b.* The 7-piece circle cut on a cylinder. *c.* An 8-piece cake using 3 cuts. *d.* An 8-piece cut on a sphere.

- 3. Three? Four?? We left this question open to think about.
- 2.2 In the circle shown below, connect A to A', B to B', and C to C' with continuous lines that don't intersect.



Struggling, we imagined a simpler variant where the positions of A' and B' were swapped. This was easily solved:



Now, since the circle can be continuously deformed (by "twisting" it) to reorient A' and B', the problem is solved.



Sean and Daniel asked the class whether this problem is still possible with more pairs of points. We again left this question open.

In both of these warmup exercises, continuous deformation was key. With that in mind, Leo presented a motto here: **Topology is geometry without shape.**

Without shape, we can study topology in spaces (sets) without distance. But this immediately has consequences. Our real analysis definition of continuity $(\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall x \text{ if } |x - a| < \delta \text{ then } |f(x) - f(a)| < \epsilon)$ depends on a notion of distance in the form of absolute value. Point set topology defines continuity without relying on it.

3 Warm-up Problem: Fundamental Theorem of Algebra

Theorem: For polynomials $p \in \mathbb{C}[x]$ (polynomials with complex coefficients) there exists a root α such that $p(\alpha) = 0$.

Proof (Idea): Consider **applying the polynomial onto a circle** C_r of radius r in \mathbb{C} - that is, the multitude of points on the complex plane (complex numbers a + bi placed at coordinates (a, b)) that lie a fixed distance r away from the origin. We are **continuously deforming the circle**.



Figure 3: Complex circle and a possible continuous deformation p = ix + 2

It was noted that:

- 1. $p(C_r)$ has to be a closed loop.
- 2. A classmate proposed the circle's image has to be symmetrical about 0, but this turned not to be true: someone proposed the polynomial p = ix + 2, which would shift the circle off center (see figure above).
- 3. Leo mentioned that while the circles' images don't have to be symmetrical about 0, it is true that the integral along the whole path is equal to 0.

Desmos link to the demonstration from class

The polynomial encoded in the demonstration is $p(x) = x^5 + (1 - 3i)x + (2 + 4i)$. With sufficiently large radius $(p \approx 4 \text{ in this})$ we see 5 strands of "circles" about the origin, reminiscent of degree 5, and the continuous deformation resembles the polynomial $p = x^5$ (since all the other terms have smaller growth order, become comparatively smaller, and affect the shape less). And with radius 0, it is of course equivalent to plugging in the point 0i + 0 into the polynomial, which gives us a constant point; in particular the constant component of p.

If this point is on the origin, the proof is complete: 0 is the aforementioned point α such that $p(\alpha) = 0$. We thus assume the constant component

of p is nonzero.

In that case, a circle with a sufficiently large radius yields something "circlish" about the origin, and a circle with radius 0 yields a point not on the origin. Thus, while shrinking the input circle toward radius 0 (effectively inputting all points within the larger radii into the polynomial) we must cross the origin. Then, the preimage of the point crossing the origin is our root α . And the proof is done, sans rigor.

We then asked the following questions:

- 1. Since the image of the big circle has winding number 5 (loops on itself 5 times, in our exemplary degree 5 polynomial), must it be true that the **curve crosses the origin 5 times** while shrinking?
- 2. How do the shapes work? How can we predict which radii will produce which shapes? (Research question)
- 3. What does "somewhere along the way" mean?

And lastly,

4. What does it mean to continuously deform?