TOPOLOGY : LECTURE 1

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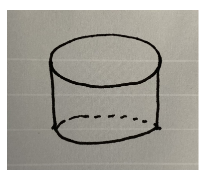
1. Introduction

After a bit of discussion about the structure and aims of the course, we started in on topology. One of the difficulties of this class is that it consists of two fairly disjoint subjects. The first, called *point set topology*, is an attempt to set up analysis in situations where one cannot measure distances (recall that basic analytic notions like convergence fundamentally depend on objects getting close to one another, so it's not at all obvious that one can define limits without referring to distance). Point set topology is a fascinating area with lots of open problems, but the name is a bit misleading—practicing topologists don't study it. (Set theorists and analysts do study problems in this area.) The second topic we'll cover is algebraic topology, which is very much what practicing topologists study. Given that one of my primary goals is to open a window into how a practicing topologist thinks, why are we spending time on point set topology? The issue is that, while the *problems* in point set topology are not of interest to topologists, the *language* of point set topology is fundamental to their work—and to many other areas of mathematics.

2. Motivating puzzles

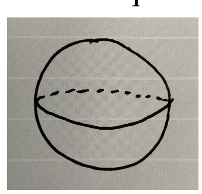
To motivate some of the underlying ideas of the class, I presented a couple of warm-up puzzles.

2.1. **Cake cutting.** Suppose you bake a chocolate cake:



A single cut through the cake divides it into two pieces. What about two cuts? As Abby pointed out, this divides the cake into four slices. [1] What about three cuts? After some thought, we all shouted out our answers simultaneously. Curiously, half the class thought the answer would be seven, while half thought it would be eight.

Before addressing this divide, we looked at a different question: what if the cake were spherical?



In this case, it becomes clear that three cuts can create eight slices: two cuts that intersect at the north and south poles, and one cut along the equator. But now observe that the same approach would divide the cylindrical cake into eight pieces as well! In other words, the problem is easier to solve in the case of a spherical cake, but squashing the sphere into a cylinder doesn't change the answer. The moral of the story is that the original geometry problem was made easier by deforming the shape (a cylinder) into a more symmetric one (a sphere), solving it in that context, and then deforming back.

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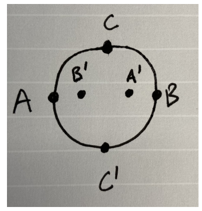
[1] Lizzie (I think?) observed that you might only get three pieces, but for now we'll focus on the maximal number of pieces you can get.

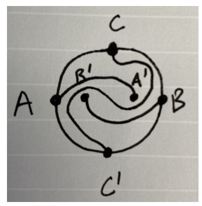
In all this, we have to be slightly careful. Emily pointed out that if we have an S-shaped cake, a single cut might create more than two pieces, even though an S-shape is a deformation of a cylinder. We'll return to this point later in the course.

2.2. **Connecting the dots.** Our second puzzle concerned the following figure:

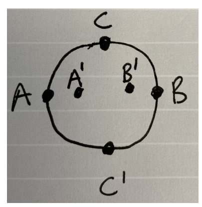


The challenge is to connect A to A', B to B', and C to C' with paths that stay in the interior of the circle and are all disjoint—no intersections! For example, if we start by connecting C and C' with a straight line, it becomes impossible to connect either of the other two pairs of points to one another.

This puzzle is surprisingly challenging, but after a bit of work some of you came up with solutions. In particular, Jacob drew a nice one on the board:



The solution to this was inspiring, but somewhat ad hoc. It turns out there's a more strategic approach. Consider the same problem as above on a different picture:



In this case, the problem is trivial to solve. But we can harness this solution to solve the original problem: rotating the points A' and B' produces Jacob's solution! Once again, rather than solving the original puzzle directly, we deform it into a simpler puzzle, solve it there, then deform back to obtain a solution to the original puzzle.

This is a familiar strategy throughout math. For example, a classic approach to solving differential equations is to apply the Fourier transform, solve the transformed equation, and then apply the inverse transform. In linear algebra, you studied similar matrices (i.e. similar transformations), which capture the same idea. For example, to reflect a point across a given line, one can rotate the line until it is horizontal, the reflect across the horizontal line (which is simple to write down), and then rotate back; the reflection across the tilted line is similar to the reflection across the horizontal one.

3. A DEEPER EXAMPLE

Our approach to both puzzles above highlights one of the key ideas of topology: some geometric features of a figure are preserved under continuous deformations, so some geometry problems are easiest to solve by neglecting the precise shape given. Our next example, one of the most famous theorems in mathematics, takes this idea and runs with it.

Theorem 3.1 (Fundamental Theorem of Algebra)**.** *Any non-constant polynomial with complex coefficients has a complex root. Alternatively, in symbols: for all* $p \in \mathbb{C}[x]$*, either* $\deg p = 0$ *or* $\exists \alpha \in \mathbb{C}$ *such that* $p(\alpha) = 0$.

To appreciate the significance of this theorem, observe that

- $x + 12 \in \mathbb{N}[x]$ but doesn't have a root in $\mathbb{N}$,

- $3x - 2 \in \mathbb{Z}[x]$ but doesn't have a root in $\mathbb{Z}$,
- $x^2 - 2 \in \mathbb{Q}[x]$ but doesn't have a root in $\mathbb{Q}$, and
- $x^2 + 1 \in \mathbb{R}[x]$ but doesn't have a root in $\mathbb{R}$.

In other words, if one begins with the most primitive type of number—the counting numbers—and starts trying to find roots of polynomials, one is forced to expand into larger and larger number systems. The Fundamental Theorem of Algebra asserts that $\mathbb{C}$ is the final stop in this chain; one will never generate new numbers outside $\mathbb{C}$ by finding roots of polynomials.[2]

The starting point of our proof of the Fundamental Theorem of Algebra is to observe that if C_r is the circle or radius r centered at 0 in the complex plane, then $p(C_r)$ must be a loop in $\mathbb{C}$. We looked at the following example, where $p(x) = x^5 + (1 - 3i)x + (2 + 4i)$. The image $p(C_4)$ looks very close to a large circle winding around 0; shrinking r to 0 makes $p(C_r)$ continuously deform down to a single point, $p(0)$. Since this deformation is continuous, there must exists some value of r for which $p(C_r)$ passes through 0. But this means that p has a root!

The same argument applies to any polynomial: given $p(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$, we can prove that $p(C_r) \approx C_{a_d r^d}$ when r is large. As $r \searrow 0$, $p(C_r)$ deforms continuously to the point $p(0)$. If $p(C_r)$ passes through 0 as $r \searrow 0$, we're done. If it doesn't, then $p(C_0)$ must 0, in which case $p(0) = 0$.

There are two major difficulties in making this proof rigorous. The first is, what do we mean by continuous deformations? We know what continuity means for functions on $\mathbb{R}$, but what does it mean for the movement of a loop consisting of uncountably many points? This is the type of question we'll tackle in point-set topology, extending notions from analysis to much more general settings. The second big issue is to prove that $p(C_r)$ must pass through 0 at some point as $r \searrow 0$. To accomplish this formally, we'll introduce a notion called the *winding number* of a loop, which measures how many times a loop wraps around 0. We'll then prove that any continuous of deformation of a loop leaves its winding number unchanged, *so long as we never pass the loop through the origin*. Finally, we'll show that the winding number of $p(C_r)$ for large r is $\deg p$, whereas for small r it's 0. Putting this all together implies that the only continuous deformations of $p(C_r)$ that never pass through the origin have $d = 0$. The study of quantities that remain invariant under continuous transformations—like the winding number—is a central theme in algebraic topology.

In sum, although the idea of the above proof is straightforward, making it rigorous will require us to develop both point-set and algebraic topology. Next class we'll start with the former goal.

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[2]Sometimes the Fundamental Theorem of Algebra is stated in a different form: that if $p \in \mathbb{C}[x]$ has degree d, then it can be factored into a product of linear polynomials in $\mathbb{C}[x]$. This is an easy consequence of our version, by induction.