Topology Notes 9/10/2024

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Summary

We began generalizing real analysis to more general spaces by first challenging our real analysis notion of distance. After setting some properties and definitions down, we worked through several examples of metric spaces remarkably different than the Euclidean one on \mathbb{R} .

1 Announcements!

- Wednesday September 18 at 1:10, the Statistics department will hold its kickoff colloquium from Ville Satopaa at Wachenheim 015: "Combining Predictions from Different Sources"!
- Monday in the evening at Goodrich's opening (9/16), Leo will host office hours. Traditional daytime office hours also started.
- **Precept meetings** will also begin soon and will have a closed-notes quiz - a proof of a theorem. David has sent more information and **logistics info/sign-up times by email**.

2 Generalizing to Metric Spaces

2.1 Defining a distance function

As said before, the purpose of point-set topology is to generalize continuity (and the capabilities of real analysis as a whole) to general spaces (think sets: what does it mean for *animals* to be continuous)? Currently, our real analysis definition of continuity fatally depends on a notion of distance. We'll later rid this requirement, as not all spaces have distance. But first, we'll challenge the concept of distance itself - it doesn't always look like \mathbb{R}^n .

What does a "notion of distance" look like? We agreed that most fundamentally, we should be able to measure how far apart two elements are: input two points from our arbitrary space X into some arbitrary distance function and receive a real number representing the distance between the two. Formally this looks like

$$d: X \times X \to \mathbb{R}$$

Next, we discussed **properties** that any function claiming to represent distance should have:

Daniel: The distance should never be negative. Barrett: d(x, y) = 0 iff x = y. Sean: d(a, b) = d(b, a) for all a, b. ??: The triangle inequality: detours can't be shortcuts! i.e.

$$d(a,c) \le d(a,b) + d(b,c)$$

Proposition: Daniel's proposal follows from the others!

Proof: By triangle inequality, $2d(a,b) = d(a,b) + d(a,b) \ge d(a,a) = 0$.

Thus, our formal definition of a metric is:

Definition. A metric space is a set X, along with a function $d : X \times X \to \mathbb{R}$ (called a metric on X), satisfying:

- 1. d(x, y) = 0 iff x = y.
- 2. d(x, y) = d(y, x).
- 3. $d(x, y) + d(y, z) \ge d(x, z)$.

2.2 Examples of Metric Spaces

Denote (X, d) for a metric space where X is a set and d is a metric (distance function) satisfying the properties above.

- 1. \mathbb{R} with respect to d(a, b) := |a b| (Euclidian metric on \mathbb{R}) It's fairly easy to verify this is a metric, since it was the basis for our intuition.
- 2. \mathbb{R}^2 with respect to $d(a,b) := \sqrt{(a_1 b_1)^2 + (a_2 b_2)^2}$ (Euclidean metric in \mathbb{R}^2)
 - There is no other real number whose square is 0 than 0; thus, if the distance between a and b is 0, the two points must share the same x coordinates and y coordinates, rendering them equal.
 - Because the square of a real number is always positive, it is clear why d(a,b) = d(b,a).
 - The triangle inequality holds: this is a nice exercise, and motivates the Cauchy-Schwarz inequality.
- 3. \mathbb{R}^2 with respect to $d(a,b) := |a_1-b_1|+|a_2-b_2|$ (Manhattan or taxi-cab metric)

Because this metric simply adds the difference in x and y values, the shortest path between two points in \mathbb{R}^2 looks like this:



This metric easily generalizes to higher dimensions.



 Now consider R[∞], a space with infinite coordinates, or the set of all sequences in R. What metrics work on this space? After Euclidean and taxi-cab metrics on finite dimensions, Isaac and Emily proposed the following:

$$d(a,b) := \left(\sum_{k=1}^{\infty} (a_k - b_k)^2\right)^{1/2} \text{ or } \sum_{k=1}^{\infty} |a_k - b_k|$$

But Michael pointed out that when these sums diverge, the distance outputs will not be in \mathbb{R} .

Alex suggested we curb the speed of growth with something like

$$\lim_{n \to \infty} \left(\sum_{k \le n} (a_k - b_k)^n \right)^{1/n},$$

but Daniel pointed out things can grow even faster regardless: $d((1, 2^{2^2}, 3^{3^3} \dots), (0, 0, 0, \dots))$ would still not converge under Alex's proposition.

Michael: Maybe we should change the question by adding an infinity point? *Lily:* Maybe we should require convergence?

Following Lily and Michael's ideas, we can restrict ourselves to just convergent sequences. Over this new space, both Emily and Isaac's ideas are valid metrics. Namely, we can consider the two spaces

$$\ell^1 := \{(a_n) \in \mathbb{R}^\infty : \sum_{n=1}^\infty a_n^2 < \infty\}$$
 and $\ell^2 := \{(a_n) \in \mathbb{R}^\infty : \sum_{n=1}^\infty |a_n| < \infty\}$

Then ℓ^1 is a metric space wrt Emily's metric, and ℓ^2 is a metric space wrt Isaac's metric. (These spaces are named after the great mathematician Lebesgue.)

- 5. \mathbb{R}^2 with respect to $d(a, b) := \max_k(|a_k b_k|)$ (Chessboard metric) This metric describes how many moves it takes a chess king to travel from one square to another.
- 6. d(a,b) = |a| + |b|, where |a| is the Euclidean distance from a to 0 (British Rail Metric)

Named after the old British railway system requiring any path between two cities to detour through London. Under this definition alone, d(x, x) = 2|x| which is not necessarily 0. Thus, we define for this metric d(x, x) := 0.

7. For a finite space such as $\{a, b, c, d\}$, a metric space can be inferred from a graph:



Such a graph with each edge having a defined length is called a **metric** graph.

8. The *p*-adic metric on \mathbb{Z} .

This metric defines the distance between integers based on their prime factorization. Under it, **pairs of integers can get arbitrarily close to one another**.

We first define the *p*-adic *norm*: a function that determines the size of a single number.

$$|n|_p := \frac{1}{p^k}$$
 where $p^k \mid n, p^{k+1} \nmid n$

In other words, the more factors of p in n, the smaller the number. If n = 0, then $|q|_n$ is defined to be 0. Then, the norm of the difference between two integers $a, b \in \mathbb{Z}$, $d(a, b) := |a - b|_p$, turns out to be a metric on \mathbb{Z} . This is possible to verify using a few tricks from number theory. It generalizes to \mathbb{Q} ; $\left|\frac{n}{m}\right|_p := \frac{|n|_p}{|m|_p}$ is a valid metric as well.

One may build \mathbb{R} by adjoining all the limits of Cauchy sequences of rationals under Euclidean distance. Interestingly, the completion of \mathbb{Q} with respect to **any** metric on \mathbb{Q} built out of an absolute value must be either \mathbb{Q} , \mathbb{R} , or \mathbb{Q}_p , the space containing all the limits of Cauchy sequences of rationals and integers under *p*-adic distance! This is called *Ostrowski's theorem*.

p-adics and *p*-adic analysis are quite relevant today, and even prove useful in solving regular, fundamental problems in \mathbb{Q} and \mathbb{Z} .

9. The discrete metric

For any nonempty space X, there exists a metric on it:

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$$

for all $x, y \in X$. It is pathological!

Next time we'll discuss what topology looks like in an arbitrary metric space.