TOPOLOGY : LECTURE 2

RAUAN KALDYBAYEV AND THEO TELLIDES

1. Introduction

Last lecture, we discussed a proof of the Fundamental Theorem of Algebra. While our presentation was (hopefully!) conceptually clear, it cannot be called rigorous, for two reasons:

- we used the concept of a "continuous deformation," which we did not define; and
- we didn't prove that the loop, at some point during its transformation from wrapping around 0 to a single point in $\mathbb{C}$, must pass through 0 somewhere.

To resolve the former, we will develop the theory of point-set topology; this is the goal of the first part of the semester, and today we're going to take our first steps towards that. (The latter goal we will resolve towards the end of the semester, based on our work on homotopy theory.)

The goal of point-set topology is to do analysis in very general settings. Real analysis deals with the real number line, a set that has a notion of distance (e.g., the distance between 3 and 8 is 5) and a notion of order (e.g., we can compare real numbers like $1 < 2$ or $5 \geq 3$). Many of the results of real analysis, for example the notion of convergence, depend on the ability to measure distance. Later in the course we will develop techniques to deal with spaces that have neither a notion of distance nor a notion of order. (Technical note: in this course, "space" can be viewed as a fancy word for "set.") This is a grand aspiration, and we will take it step-by-step. We start off by discussing sets that have a notion of distance, but not necessarily an order; such sets are called *metric spaces*. Metric spaces are the main focus of the next few lectures.

2. What is a metric space?

Informally, a metric space is a set where we can measure distance between elements. For two elements x, y of this set, the distance between x and y is given by $d(x, y)$, where d is a function called the *metric*. The metric determines which points are close to each other and which are far.

We have some intuitive understanding of what we want from a reasonable measure of distance. For example, distance should always be a non-negative real number; the distance between two points shouldn't depend on which order I give you the points; the only way the distance between two points is 0 is if they're secretly the same point; and detours should never be shortcuts, i.e. any distance function should produce the *shortest* distance between two points. We now formalize these conditions:

Definition. A *metric* on a set X is any function $d : X \times X \to \mathbb{R}$ satisfying all of the following:

(1) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(2) $d(x, y) = 0$ if and only if $x = y$;
(3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

We've captured all of our intuitive notions about distance, save one: nowhere in the above did we specify that distance must be non-negative. It turns out this is implicit in our definition:

Proposition 2.1. *If d is a metric on X, then $d(x, y) \geq 0$ for all $x, y \in X$.*

Proof. We have

$$0 = d(x, x) \leq d(x, y) + d(y, x) = 2d(x, y)$$

from which the claim follows. Note that we used all three properties of the metric in this proof! $\square$

Date: September 13, 2022.
Based on a lecture by Leo Goldmakher.

With this, we can define a fundamental concept:

Definition. If d is a metric on a nonempty set X, we call the pair (X, d) a **metric space**.

To get a better feel for this definition, we explore a bunch of examples—some familiar, some less so.

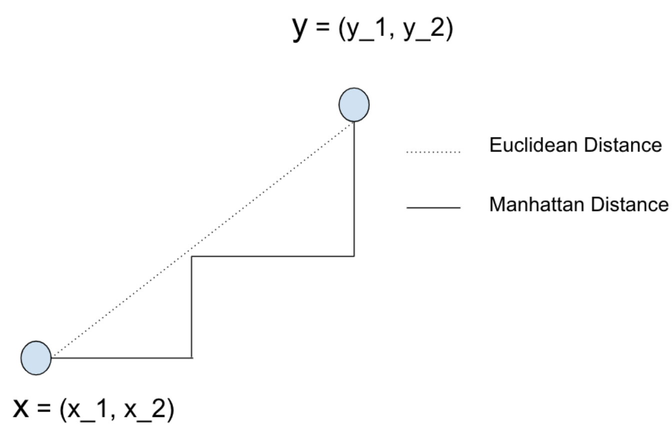
3. Examples of metric spaces

Here we list some examples of metric spaces; proofs are left to the reader.

(1) The most familiar metric space is $\mathbb{R}$ with the *Euclidean metric* $d(x, y) = |x - y|$.

(2) The plane $\mathbb{R}^2$ with the Euclidean metric $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

(3) The plane $\mathbb{R}^2$ with the "taxicab metric" (aka the Manhattan metric) $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$. The motivation for this metric is as follows. If you live in city whose streets form a grid and you want to walk from point A to point B, the distance you would have to walk is given by the taxicab metric.



Comparison of Euclidean vs taxicab metric in $\mathbb{R}^2$

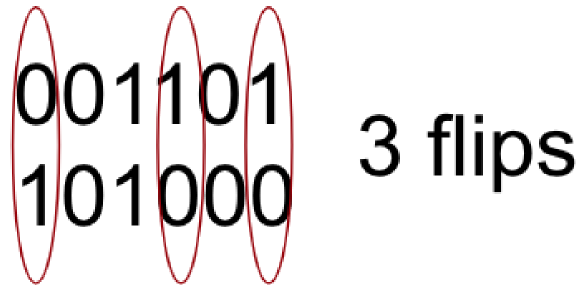
(4) The plane $\mathbb{R}^2$ with the "chessboard metric" $d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$. One way to think about this metric is that it represents the number of steps it takes the king to walk from x to y. (Recall that a king can move one square at a time in any of the 8 directions available, including the diagonals.)

(5) The plane $\mathbb{R}^2$ with the "British rail metric"

$$d(x, y) = \begin{cases} 0 & x = y \\ |x| + |y| & x \neq y \end{cases}$$

The motivation for this metric is as follows. In the past, it used to be that when traveling between any two cities in Britain by railway, one would have to travel via London; thus, the distance from Liverpool to Manchester was measured by the Euclidean distance from Liverpool to London and the from London to Manchester. Of course, to travel from Liverpool to itself one doesn't have to pass through London, so the distance is 0.

(6) The set of natural numbers $\mathbb{N}$ (starting with zero!) with the Hamming distance $d(m, n)$ being the number of places where the binary representations of m and n disagree. For instance, consider $d(13, 40)$. The binary representation of 40 is 101000, and the binary representation of 13 is 1101, which is equivalent to 001101. The strings 101000 and 001101 differ in 3 places, so $d(13, 40) = 3$.



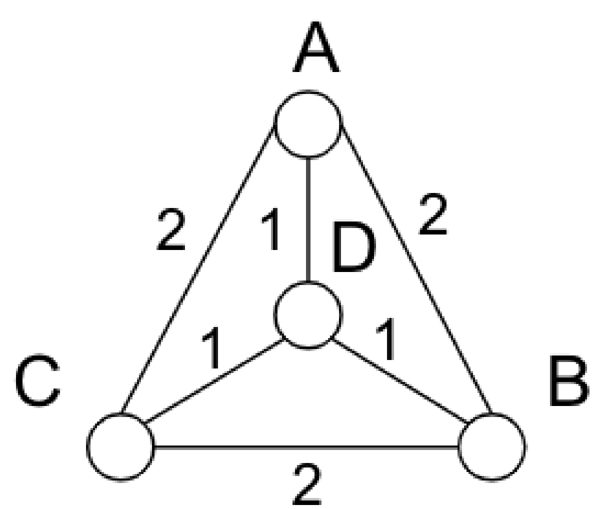
Hamming distance example

(7) The integers $\mathbb{Z}$ with the 7-adic distance $d(x,y) = |x-y|_7$. Here, $|n|_7$ is defined as follows. A nonzero integer n can be uniquely written as $7^a b$ for some integers a, b, where b is not divisible by 7. We set $|n|_7$ to 7^{-a}. Thus, for example, $d(2,8) = 1$, $d(2,9) = 1/7$, and $d(2,100) = 1/49$. Note that the more divisible by 7 a number is, the smaller its 7-adic absolute value. This suggests setting $|0|_7 := 0$, since 0 is divisible by arbitrary large powers of 7 and hence $|0|_7$ should be arbitrarily small.

It turns out that the 7-adic distance extends to a metric on $\mathbb{Q}$: simply set $|m/n|_7 := |m|_7/|n|_7$. Recall from real analysis that if we can construct $\mathbb{R}$ from $\mathbb{Q}$ by completing $\mathbb{Q}$ with respect to the Euclidean metric. What happens if we try to complete $\mathbb{Q}$ with respect to the 7-adic metric instead? We get a rather different set of numbers, called the 7-adic numbers and denoted $\mathbb{Q}_7$. This space has the same cardinality as $\mathbb{R}$, and of course both $\mathbb{Q}_7$ and $\mathbb{R}$ contain $\mathbb{Q}$, but the similarities end there. For example, in $\mathbb{R}$ there is precisely one solution to the equation $x^3 = 6$, while in $\mathbb{Q}_7$ there are three distinct solutions, so $\mathbb{Q}_7$ contains numbers that $\mathbb{R}$ does not. By contrast, $x^2 = 7$ has two solutions in $\mathbb{R}$ but none in $\mathbb{Q}_7$, so $\mathbb{R}$ contains numbers $\mathbb{Q}_7$ does not.

More generally, for any prime p one can complete $\mathbb{Q}$ with respect to a p-adic metric to form the set of p-adic numbers $\mathbb{Q}_p$. The p-adic metric and the idea of forming $\mathbb{Q}_p$ might seem artificial, but in fact these spaces play a very important role as the only completions of $\mathbb{Q}$ other than $\mathbb{R}$. Somewhat more precisely, a remarkable theorem of Ostrowski asserts that given any absolute value on $\mathbb{Q}$, the completion of $\mathbb{Q}$ with respect to it is either $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{Q}_p$ for some p.

(8) The weighted graph below consisting of points $\{A, B, C, D\}$ with $d(x,y)$ being the weight on the edge connecting x and y (and 0 if $x = y$) is a metric space.



Metric space on a finite set of points with predetermined distances between points

(9) We can make an arbitrary nonempty set X into a metric space by using the "discrete metric"

$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

For example, if X is the set of states in the US, then $d(\mathrm{MA}, \mathrm{MA}) = 0$ while $d(\mathrm{MA}, \mathrm{NY}) = 1$.

4. Topology in metric spaces

Some of the most familiar sets from real analysis are open and closed sets. How do these generalize to metric spaces? The most basic example of an open set in $\mathbb{R}$ is an open interval (a, b), but this doesn't generalize nicely to metric spaces since it relies on having a notion of order (which a metric space might not have). Staring at some pictures in $\mathbb{R}^2$ (with the usual Euclidean metric), we quickly decided that a set is open iff it doesn't contain its boundary, and closed iff it does. But what is a boundary, exactly?

An initial proposal was that a point is on the boundary of a set A iff when you step in one direction you land in A, while if you step in the other direction you land in A^c. What is a 'step'? And what do we mean by 'direction'? Indeed, as stated, this applies to literally any point in the space—just take a big enough step! Thus we restrict ourselves to small steps: a point is on the boundary of A iff an arbitrarily small step in one direction always lands us in A, and an arbitrarily small step in the other direction always lands us in A^c. This doesn't resolve the issue of what direction means in an arbitrary metric space; we will take this up next class.