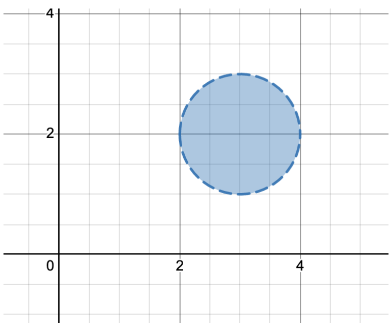
Lecture Notes, September 15, 2022

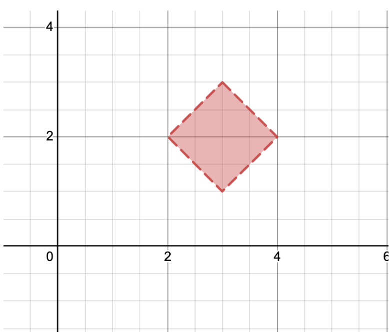
Our goal is to formalize the notions of open and closed sets, which relies on some other definitions first:

Definition 1. *Given a set* X *and* $\alpha \in X$*, for a real number* $r \in \mathbb{R}$ *we define the* ***open ball*** *of radius* r *around* α *to be* $\mathscr{B}_r(\alpha) = \{x \in X | d(\alpha, x) < r\}$

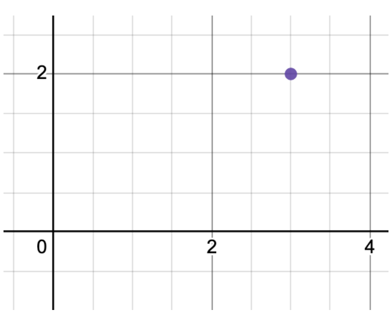
Example 1. *Graph* $\mathscr{B}_1((3,2))$ *in* $\mathbb{R}^2$ *w.r.t the Euclidean metric.*



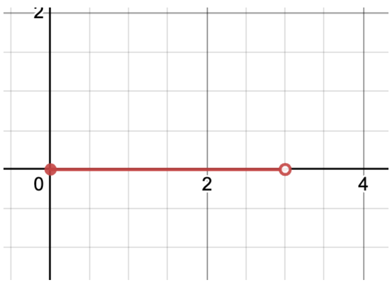
Example 2. *Graph* $\mathscr{B}_1((3,2))$ *in* $\mathbb{R}^2$ *w.r.t the taxicab metric.*



Example 3. *Graph* $\mathscr{B}_1((3,2))$ *in* $\mathbb{R}^2$ *w.r.t the discrete metric.*



Example 4. *Graph* $\mathscr{B}_2(1)$ *in* $\mathbb{R}_{\geq 0}$ *w.r.t the Euclidean metric.*



We formalize the notion of boundary as well:

Definition 2. *For a metric space X and a subset $A \subseteq X$, $a \in X$ is on the **boundary** of A if $\forall \delta > 0, \mathscr{B}_\delta(a) \cap A \neq \emptyset$.*

The boundary of a set A is denoted ∂A and is the collection of all points $a \in X$ such that a is on the boundary of A.

Definition 3. *A set is **open** if $\partial A \cap A \neq \emptyset$.*

Definition 4. *A set is **closed** if $\partial A \subseteq A$.*

Many sets are neither open nor closed, and some sets are both. A set that is both open and closed is called **clopen**

Proposition: A is open iff $\forall \alpha \in A$ there exists some $\delta > 0$ such that $\mathscr{B}_\delta(\alpha) \subseteq A$.

We say a point α that satisfies this condition is an interior point of A and denote the set of interior points of A as $\text{int}(A)$.

Note: The empty set is clopen, as is its complement. However, these are not the only clopen sets.

Claim. *$\mathscr{B}_r(a)$ is open.*

Proof. Homework. □

Some examples:

- the open interval $(1, 2)$ in $\mathbb{R}$ with respect to the Euclidean metric is open.
- the interval $(1, 2) \times 1$ in $\mathbb{R}^2$ (an open interval in $\mathbb{R}^2$ at y-value 1) is not open. It is also not closed.
- In $\mathbb{R}_{\geq 0}$ with respect to the Euclidean metric, [0,3) is open. One way to see why is by observing that $\{3\}$ is the entire boundary.