Topology Notes 9/12/2024

Williams College, Math 374

Summary/Topics

We defined metric balls, which we used to define closed and open sets on any metric space..

1 More on p-adics

Last time, we defined the *p*-adic norm $|\cdot|_p$ in the following way:

$$|n|_p := \begin{cases} 0 & \text{if } n = 0, \\ p^{-k} & \text{if } n \neq 0, \text{ where } p^k \mid n \text{ and } p^{k+1} \nmid n \end{cases}$$

It may seem odd to define 0 separately, but it is in fact intuitive: 0 is infinitely divisible by p, and $\frac{1}{p^{\infty}} \to 0$.

Intuitively, *p*-adics compare the *end* of a number, whereas Euclidean distance compares the start. (In other words, the Euclidean distance is intuitive if we compare numbers right to left, whereas the *p*-adic distance is intuitive if we compare numbers left to right.) For instance, 351 and 352 are close under Euclidean distance, but far under 10-adic distance; meanwhile, 351 and 451 are far Euclidean-wise but close 10-adically ($\frac{1}{100}$ apart). This idea extends to the *p*-adic metric .

In fact, *p*-adics care so little about the start of a number that *p*-adic numbers may have infinite digits trailing to the left of a number, just as under the Euclidean distance we can have numbers the extend infinitely far to the right. For example, with respect to the Euclidean metric, we have 0.999... = 1, whereas with respect to the 5-adic metric, ...444 = -1. (More "intuitively", with respect to the 10-adic metric, ...999 = -1.)

2 Topology of Metric Spaces

We discuss what open and closed sets look like on arbitrary metric spaces.

1. We're familiar with open intervals in \mathbb{R} , which look like

$$(a,b) := \{x \in \mathbb{R} : a < x < b\}$$

Closed intervals look like

$$[a,b] := \{ x \in \mathbb{R} : a \le x \le b \}.$$

Intervals that are "half open, half closed" are said to be neither.

2. In \mathbb{R}^2 , we have a similar notion:



We concluded from the above picture that any space X and a subspace $A \subseteq X$, A is **open iff it contains none of its boundary.** Serahn observed that a subspace $B \subseteq X$ is **closed iff it contains all of its boundary**.

This is all great, but what on earth is *the boundary* of a space?

Daniel and Leo agreed that a point on the boundary should be very close to points on the inside and the outside; no matter how far you zoom into a point on the boundary, there will always be points of A and points of A^c visible.

Lily formalized this: If you made a ball on the boundary, no matter how small the ball was, it would contains points both in and outside the subspace. In other words:

Definition. An open ball of radius r centered around point p is the set of all point that are within a distance r of p. More precisely:

$$B_r(p) := \{ x \in X : d(x, p) < r \}.$$

We noted the positive radius here acts as an analogue to " $\varepsilon > 0$ " in real analysis.

2.1 Examples of Open Balls in Different Metrics

1. Of course, an open ball in Euclidean \mathbb{R}^2 looks as expected:



2. We found a ball in \mathbb{R}^2 under the taxi-cab metric looks like a diamond:



3. Under the discrete metric, $B_r(p) = \{p\}$ for any $r \leq 1$ and $B_r(p) = \mathbb{R}^2$ for any r > 1. Note that in *any* metric space, a ball of positive radius must contain



its center. (Why?) Thus, these discrete open balls are the most extreme examples.

4. In \mathbb{R} with respect to the Euclidean metric, $B_2(1) = (-1, 3)$:



5. However, in $\mathbb{R}_{\geq 0}$ with respect to the Euclidean metric, $B_2(1) = [0,3)$:



Now that we have a rigorous notion of "zooming in", we can make precise Lily's proposed definition of boundary:

Definition. Given a metric space (X, d) and a subspace $A \subseteq X$, p is on the boundary of A (denoted ∂A) iff $\forall \varepsilon > 0$ we have

$$B_{\varepsilon}(p) \cap A \neq \emptyset$$
 and $B_{\varepsilon}(p) \cap A^{c} \neq \emptyset$

That is, every ball centered at a point on ∂A contains points of A and of A^c .

Remark. It's not an accident that the same symbol ∂ is used to denote the boundary and the derivative. For example, the derivative of the area of a circle is the length of the boundary, and the derivative of the volume of a sphere is the surface area of the boundary. More generally, integrating the derivative of a function over a region is the same as integrating the function itself over the boundary of the region.

Rewriting our initial formulation of open and closed, we have:

Definition. A is open iff $\partial A \cap A = \emptyset$, and closed iff $\partial A \subseteq A$.

Note that we are still dependent on distance: openness depends on boundaries, which depend on open balls, which depend on distance.

Example. The interval (2,3) is open in \mathbb{R} . But (2,3) \times {1} isn't open in \mathbb{R}^2 !



The points on the line in between 2 and 3 are on the boundary, and yet in the set. However, it isn't closed either; (2, 1) and (3, 1) are boundary points, but not in the set. It is **neither open nor closed!**

Here, Lily asked whether something can be **closed and open at once**.

Michael pointed out that \emptyset is, for silly reasons: \emptyset contains nothing, and thus none of its boundary. But it has no boundary, so it contains all of its boundary. Also, in \mathbb{R} , \mathbb{R} itself is closed and open, again for silly reasons.

In Euclidean \mathbb{R}^2 , these are the only sets which are closed *and* open. However, this is not always the case; in fact, this phenomenon occurs so frequently in some spaces that it has earned its own terrible portmanteau: **clopen**.

Lastly, we examined the following:

Proposition. For any metric space (X, d) and any $p \in X$, the singleton $\{p\}$ is closed.

We came up with some ideas, and we'll start next class with a rigorous proof.