Topology Notes 9/17/2024

Williams College, Math 374

Summary/Topics

We discussed different ways to think about and work with open and closed sets on metric spaces. We discussed several properties of both, and decided open sets are easier to work with.

1 Announcements!

- Friday September 20th at 1:00, Prof. Susan Loepp will give a faculty seminar on prime ideal structures in Wach 015.
- Precepts are happening, and **Problem Set 2 is due** this Thursday 9/18.

2 Is $\{p\}$ closed?

Recall from last time that in a metric space (X, d), a subset A is A is open iff $\partial A \cap A = \emptyset$, and closed iff $\partial A \subseteq A$.

Also recall that a point p is on the boundary of A iff $\forall \varepsilon > 0$ we have

 $B_{\varepsilon}(p) \cap A \neq \emptyset$ and $B_{\varepsilon}(p) \cap A^{c} \neq \emptyset$

Last time, we examined the following question:

Proposition 1. For any metric space (X, d) and any $p \in X$, the singleton $\{p\}$ is closed.

We found that any point $x \in \partial\{p\}$ had to be infinitesimally close to p, as any ball $B_{\varepsilon}(x)$ of radius ε had to contain p. We wanted $\partial\{p\} = \{p\}$.

This time, we encountered a hidden subtlety: if there is a radius r > 0 such that $B_r(p)$ contains nothing but $\{p\}$ (like balls of radius 0 < r < 1 under the discrete metric), then there are **no points infinitesimally close to both** $\{p\}$ and $\{p\}^c$. In this case, $\partial\{p\}$ is empty, and p is called an *isolated point*.



Figure 1: A lonely isolated point with an open ball that contains nothing but itself

Otherwise, because there cannot be any $x \in \partial\{p\}$ where $d(x, p) \neq 0 \leftrightarrow x \neq p$, we do have that $\partial\{p\} = \{p\}$. Here, since by negation we have $\forall \varepsilon > 0$, $B_{\varepsilon}(p) \cap \{p\}^c \neq \emptyset$, p is called a **limit point**. (Sometimes this is also called a *cluster point* or an *accumulation point*.)

In either case, $\partial\{p\} \subseteq p$, so $\{p\}$ is closed. \Box

3 Topology of Metric Spaces, cont.

The problem set this week discussed other, equivalent ways to think about "closedness" and "openness." For instance:

Proposition 2. Under a metric space (X, d), and a subspace $A \subseteq X$, A is open iff any point p of A is an **interior point**: $\exists \varepsilon$ such that $B_{\varepsilon}(p) \subseteq A$. A is closed if it contains all its **limit points**: \exists a sequence (a_n) in A, each $a_n \neq p$, such that $d(a_n, p)$ can approach 0.

Remark. Both notions of limit points are equivalent: take any sequence of radii ε approaching 0, and then any sequence of points inside each radius.

Another very useful property on the problem set:

Proposition 3. A is open iff A^c is closed.

Again, things can still be open and closed at the same time, or neither open nor closed. It's very useful to be fluent with all these different perspectives on closedness and openness. For example, here's one immediate consequence:

Proposition 4. Unions of open sets are open.

Lily justified this: All points in unions of open sets remain interior points. Then Alex noted by Proposition 3 intersections of closed sets are closed.

He then asked whether **any** open set [with at least two points] can be written as a union of [nonempty] open sets. This turned out to be true: in an open set O, every point is an interior point, meaning an open ball around it with small enough radius would be contained in O. Thus, we can take the union of open balls around every point in O.

Remark. (Daniel) The union of open intervals is open even if the union is not finite!

Proposition 5. Are the intersections of open sets always open?

No. Take for instance in \mathbb{R} under the Euclidean metric

$$\bigcap_{\varepsilon > 0} (-\varepsilon, \varepsilon) = \{0\}$$

which is not open! However, **finite** intersections of open sets are always open.

4 A general principle

Given Proposition 3, understanding the open sets in a space is equivalent to understanding the closed sets. In practice, however, there's an asymmetry:

open sets are easier to understand than closed sets.

Here's one illustration of this. What does a typical open set in \mathbb{R} (wrt the Euclidean metric) look like? The most obvious type of open set is an open interval. The next most obvious type of open set is a union of two (or more) open intervals. What other types of open sets are there?

Proposition 6. Every open set in \mathbb{R} under the Euclidean metric is a countable union of open intervals.

Proof sketch. Pick an open set $\mathcal{O} \subseteq \mathbb{R}$. We know that every point of \mathcal{O} is an interior point; in other words, for any $p \in \mathcal{O}$, there exists an open interval I_p such that $p \in I_p \subseteq \mathcal{O}$. We instantly deduce from this that \mathcal{O} is a union of open intervals:

$$\mathcal{O} = \bigcup_{p \in \mathcal{O}} I_p. \tag{1}$$

This almost proves the proposition, except that the union above might not be countable. To express \mathcal{O} as a countable union of open intervals, we must work a bit harder:

- 1. "Inflate each interval with air"—take each I_p to be maximal, by redefining each I_p to the union of all open intervals containing p that are subsets of \mathcal{O} .
- 2. It's an exercise to verify that $I_p = I_q$ or $I_p \cap I_q = \emptyset$ for any $p, q \in \mathcal{O}$.
- 3. Drop all redundancies in the union (1): we can choose a set $S \subseteq \mathcal{O}$ such that

$$\mathcal{O} = \bigcup_{p \in \mathcal{O}} I_p = \bigsqcup_{p \in S} I_p$$

(Here \sqcup is the *disjoint union*, i.e. a union of sets that are all pairwise disjoint.)

4. S must be a countable set! Why? Michael pointed out that each nonempty open interval contains a rational (" \mathbb{Q} is dense in \mathbb{R} "). In particular, if S were uncountable, our disjoint union would be composed of uncountably many disjoint open intervals, each containing a rational—which would imply the existence of uncountably many rationals!

We've thus expressed \mathcal{O} as a countable union of open intervals. Since \mathcal{O} was an arbitrary open set, we've proved the claim.

The above proposition demonstrates that the open sets of \mathbb{R} are easy to characterize. What about the closed sets? Alex pointed out that we can use the proposition to give an easy description: a subset of \mathbb{R} is closed iff its complement is a union open intervals. Is there a way to characterize closed sets directly, without going through the opens? It's not obvious. Indeed, there exist some very strange closed sets! For example, recall the **Cantor Set.** This is constructed by starting with the closed unit interval, removing the middle third of it (leaving two closed intervals), removing the middle thirds of each of these (leaving four closed intervals), etc. Here's an illustration:

The **Cantor set** is what's left after all iterations of this process; more rigorously, it's the intersection of all of the C_i . Since each iteration is a closed set, and intersections of closed sets are closed, the Cantor Set is closed. However, it turns out to have some truly bizarre properties. For example, it's uncountable but has measure 0 and contains no intervals! You'll explore the Cantor set further in your problem set next week.