TOPOLOGY : LECTURE 4

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1. RECAP

Last time: Given metric space (X, d), $A \subseteq X$ is open iff $\partial A \cap A = \emptyset$. Equivalently, A is open if every point in A is in the interior of A, so there is an open ball in A which contains the point.

$A \subseteq X$ is closed iff $\partial A \subseteq A$.

$x \in \partial A$ iff $\forall \delta > 0$, $B_\delta(x) \cap A \neq \emptyset$ and $B_\delta(x) \cap A^c \neq \emptyset$.

$x \in$ int A iff $\exists \delta > 0$ s.t $B_\delta(x) \subseteq A$.

1.1. Open sets in $\mathbb{R}$ (with Euclidean metric).

All open intervals (a, b), where $a < b$, are open sets in $\mathbb{R}$. $\emptyset$ and $\mathbb{R}$ are also open, we can think of these as the intervals $(0, 0)$ and $(-\infty, \infty)$. We can also take the union of any finite number of open intervals to get a new open set.

Is a single point such as $A = \{5\}$ open? We have that $5 \in \partial A$, and $5 \in A$, so $\partial A \cap A \neq \emptyset$ and A is not open.

In fact, we have that every open set in $\mathbb{R}$ with this metric is the union of intervals:

Theorem 1.1. *All open sets in $\mathbb{R}$ with respect to the Euclidean metric are countable unions of open intervals.*

Proof sketch. Given $A \subseteq \mathbb{R}$ open, pick some $x \in A$. Then there exists an open interval containing x which is contained in A. Thus, there exists a maximal interval I_x containing x which is contained in A. "Maximal" means that I_x contains all other such intervals, we can quickly construct I_x by taking the union of all such intervals. Then we have that

$$A = \bigcup_{x \in A} I_x$$

since each I_x contains x and each I_x is contained in A. We can remove the "superfluous" x's to get a disjoint union. This step needs justification but this is just a proof sketch.

Now, we have expressed A as a disjoint union of open intervals. Why must there only be countably many such I_x?

Each interval contains a rational, and there are only countably many rationals, so this union is countable. □

The above result does not hold for closed sets, as we can find a countable union of closed sets which is not closed:

$$[0, 1/2] \cup [1/2, 3/4] \cup [3/4, 7/8] \cup \cdots = [0, 1) \tag{1.1}$$

which is not closed. Where does the proof break down when we replace open with closed? - this is left as an exercise. However, we can characterize closed sets with the following result:

Proposition 1.2. *A is closed iff A^c is open.*

Proof.

$$\begin{aligned} A \text{ is closed} &\iff \partial A \subseteq A \\ &\iff \partial A \cap A^c = \emptyset \\ &\iff \partial(A^c) \cap A^c = \emptyset \\ &\iff A^c \text{is open.} \end{aligned}$$

We need to show that $\partial A = \partial(A^c)$, but we can do this later. □

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Remark. The above result holds in any metric space. In general, we are talking about metric spaces right now because they can be seen as an "easy" version of general topological spaces, or at least a more grounded version. We will begin working with general topological spaces soon, and many of these same results will hold.

1.2. **Convergence of sequences.** Given some sequence $(a_n) \subseteq X$, where (X, d) is a metric space.

Definition. (a_n) converges iff $\exists L \in X$ s.t $\forall \epsilon > 0$, there exists N s.t if $n > N$ then $d(a_n, L) < \epsilon$.

In other words, (a_n) gets arbitrarily close to L.

Example 1. Does $(a_n) = (1/n)$ converge?

(1) In $\mathbb{R}$ with the usual Euclidean metric, **yes**.
(2) In $\mathbb{R} - 0$ with the usual Euclidean metric, **no**. While in some sense the sequence gets closer and closer to 0, 0 is not in the space so it does not get closer to any point.
(3) In $\mathbb{R}$ with the discrete metric, **no**. $1/n$ will always be a distance of 1 away from 0, so it will never converge to 0 (or anything else).

Pick $A \subseteq X$, and suppose $(a_n) \subseteq A$. If (a_n) converges (to some point in X), must $\lim_{n\to\infty} a_n \in A$?
No: for example, if $A = (0, 1]$, then $\lim 1/n = 0 \notin A$ but $1/n$ is always in A.
So where can the limit live if it is not in A? First, a definition.

Definition. Set $\overline{A} = A \cup \partial A$. $\overline{A}$ is called the **closure** of A.

Remark. The closure of A consists of all points close to A. Any point which is greater than 0 distance away from all points in A is not in the closure, which we can make precise with the following statement.

Proposition 1.3. *For all $x \notin \overline{A}$, $\exists \delta > 0$ s.t $d(x, a) \geq \delta$ $\forall a \in A$.*

Proof. Pick some $x \notin \overline{A}$. Then $x \notin A$ and $x \notin \partial A$. Then there exists $\delta > 0$ such that $B_\delta(x) \cap A = \emptyset$ or $B_\delta(x) \cap A^c = \emptyset$. But $x \notin A$, so $x \in A^c$ and we definitely have that $B_\delta(x) \cap A^c \neq \emptyset$. This means we must have $B_\delta(x) \cap A = \emptyset$. Thus for all $a \in A$, we have that $a \notin B_\delta(x)$, so $d(a, x) \geq \delta$. □

We immediately get the following result as a corollary.

Corollary 1.4. *If $(a_n) \subseteq A$ converges, then $\lim a_n \in \overline{A}$.*

Proof. If (a_n) were to converge to a point L outside of $\overline{A}$, then $d(L, a_n) \geq \delta$ for some δ by the previous proposition, which is a contradiction on the definition of a limit. □

Corollary 1.5. *We have that $X = \overline{A} \sqcup \text{int}\,(A^c)$. Here $\sqcup$ means disjoint union. Note also that $\text{int}\,(A^c)$ denotes the interior of A^c.*

Proof. This follows from our earlier proposition. If something is not in the closure of A, it is at least δ from everything in A, so it is in the interior of A^c. □

Remark. $\overline{A} = \text{int}\, A \sqcup \partial A$. Thus the entire metric space is $X = \text{int}\, A \sqcup \partial A \sqcup \text{int}\, A^c$. This shows that the boundary of A is the same as the boundary of A^c if we simply swap A and A^c.