Motivation for Topological Spaces

Given $X \neq \emptyset$ without a metric, we have a space without a notion of distance. One can impose any metric on this set, but it won't necessarily preserve the desired geometry (Recall that we can put the discrete metric on any nonempty space, but that metric doesn't yield any information about the space's geometry!). *We want to be able to replicate ideas from Real Analysis without a notion of distance between arbitrary points in the space.*

- We want to understand $\lim_{n\to\infty} a_n = L$. In Real Analysis, this means that a_n "gets close and stays close to L"
- f is continuous at α, in Real Analysis, this meant points in neighborhoods around α were staying close to $f(\alpha)$.
- We need to define what it means for a set to be open or closed
  - We can't use open balls because those rely on a metric
  - How could you define the boundary of a set without distance?

In order to resolve many of the problems with spaces without metrics, we can use a concept of *closeness* that we have already developed, namely closure. Last class we rigorously showed that the closure of a set A is the set of all points close to A, so we can use this idea.

Closure of a set

We need to define what a closure is without a metric. Given $X \neq \emptyset, A \subseteq X$, what is $\overline{A}$? We could use $A \cup \partial A$, but then we would need to define the boundary first. We could also define $\overline{A} = \bigcap_{X\in\Gamma(A)} X$ where $\Gamma(A)$ is the set of all closed sets containing A. But then we would need to define a closed set first. So we will define the properties that we want the closure to have as a motivation for its definition:

Definition 0.1. A **closure operator** on a space X is a function $\overline{\cdot} : \mathscr{P}(X) \to \mathscr{P}(x)$ with the following properties

(i) $A \subseteq \overline{A}$

(ii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

(iii) $\emptyset = \overline{\emptyset}$

(iv) $\overline{A} = \overline{\overline{A}}$ for any set A

(v) $X = \overline{X}$

(vi) $\overline{A \cap B} = \overline{A} \cap \overline{B}$

Note that property (vi) is not actually true. For example $\overline{Q} = \mathbb{R}$, and $\overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}$, so $\overline{\mathbb{Q}} \cup \overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}$. But $\mathbb{Q} \cup \mathbb{R} \setminus \mathbb{Q} = \emptyset$, so $\overline{\mathbb{Q} \cup \mathbb{R} \setminus \mathbb{Q}} = \emptyset$.
Additionally, although property (v) is true, it follows as a consequence of property (i). So it is redundant.
Therefore we will define the closure operator to have the first four properties.

Closed sets

Definition 0.2. A set A is **closed** if and only if $A = \overline{A}$

Proposition 0.3. Given a closure operator on a set X the following hold:

(i) X and $\emptyset$ are closed.

(ii) finite unions of closed sets are closed.

(iii) Arbitrary intersections of closed sets are closed.

Proof.

(i) This is by the definition of closure.

(ii) Given two closed sets A, B, by def $A = \overline{A}$, and $B = \overline{B}$. So $A \cup B = \overline{A} \cup \overline{B} = \overline{A \cup B}$

Now given three closed sets A, B, C we have $A \cup B \cup C = (A \cup B) \cup C = \overline{(A \cup B)} \cup \overline{C} = \overline{A \cup B \cup C}$.

So by induction, all finite unions of closed sets are closed. Note that this does not work for infinite unions because induction can only be used to prove finite cases.

(iii) Given a collection of closed sets $\{C_\alpha\}$ we want to show that $\overline{\bigcap_\alpha C_\alpha} = \bigcap_\alpha C_\alpha$.

Now by (i) $\bigcap_\alpha C_\alpha \subseteq \overline{\bigcap_\alpha C_\alpha}$, so we need to show containment the other way.

So let $C_\beta \in \{C_\alpha\}$, we have that $\bigcap_\alpha C_\alpha \subseteq C_\beta$. Taking the closure of both sides gives $\overline{\bigcap_\alpha C_\alpha} \subseteq \overline{C_\beta} = C_\beta$. And since this is true for ant $C_\beta \in \{C_\alpha\}$, we have that $\overline{\bigcap_\alpha C_\alpha} \subseteq \bigcap_\alpha C_\alpha$

Note that we used the fact that if $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$, which will be a homework exercise next week.

□

Given a notion of *closure*, we defined *closed*. Can we go in reverse? In other words, if I defined "closed", how would you define closure? It was noted we can set $\overline{A} := \bigcap_{C \in \Gamma(A)} C$ where $\Gamma(A)$ indexes all closed sets containing A. Essentially we have set $\overline{A}$ to be the smallest closed set containing A. These two definitions lead to an equivalent definition of closure and closed sets.

Open sets

Now we will use our definition of closed sets as a means of defining open.

Definition 0.4. A is **open** $\iff A^c$ is closed

Open sets have the following properties;

(i) $\emptyset$ and $\mathbb{R}$ are open sets

(ii) Infinite intersections of open sets are open

(iii) Arbitrary unions of open sets are open

Defining Topological Spaces

Definition 0.5. A **Topology** on X is any collection of subsets of X satisfying (i) - (iii) defined above for open sets.

Note: there is actually nothing special about defining our topology in terms of closed sets instead of using open sets, but by convention we will present topologies in terms of open sets here.
Now we will present examples. Using different metrics on the same space, we will demonstrate that the same space will have different topologies under different metrics.

(I) What is the topology of $\mathbb{R}$ with respect to the discrete metric? Meaning, what are the open sets of $\mathbb{R}$ if the metric is the discrete metric?

The answer is that the open sets are $\mathscr{P}(\mathbb{R})$. You can verify this fact by observing that for any $x \in \mathbb{R}, \mathcal{B}_1(x) = \{x\}$, so any isolated element is an open set, and arbitrary unions of open sets are open. Notably also $\varnothing$ must also be an open set (it does contain its boundary in this metric).

(II) What is the topology of $\mathbb{R}$ with respect to the standard Euclidean metric?

As we discussed last class, the open sets in this topology are

$\{$countable unions of open intervals$\}$, since any open set can be reduced to a countable union of open intervals, this is a more concise definition than the set of arbitrary open intervals.

(i) Why is this distinct from the topology induced by the discrete metric?

Notice that $\{1\} \in \mathscr{P}(\mathbb{R})$, so in the topology induced by the discrete metric, $\{1\}$ is an open set, but under the Euclidean metric, $\partial(\{1\}) = \{1\}$, so $\{1\}$ contains its boundary and is therefore a closed set, so they are not equivalent topologies!