TOPOLOGY: LECTURE 6

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1. Topological Spaces

Definition (topological space). A topological space is a pair $(X, \mathcal{T})$ that consists of a set $X \neq \emptyset$ and a topology $\mathcal{T}$ on X, i.e. $\mathcal{T}$ consists of subsets of X such that:

(i) $\emptyset, X \in \mathcal{T}$
(ii) $\forall A, B \in \mathcal{T}, A \cap B \in \mathcal{T}$
(iii) T is closed under union

2. Examples of Topological Spaces

1. $\mathbb{R}_{\text{usual}}$ is $(\mathbb{R}, \mathcal{T}_{\text{usual}})$ where $\mathcal{T}_{\text{usual}} = \{\mathcal{O} \subseteq \mathbb{R} : \mathcal{O} = \text{open with respect to the Euclidean metric on } \mathbb{R}\}$.

2. For $X = \{1, 2, \text{Alice}\}$, some topologies are:

- $\mathcal{T} = \{\emptyset, X\}$
- $\mathcal{T} = \mathcal{P}(X)$
- $\mathcal{T} = \{\emptyset, X, \{1\}\}$

The set $\{\emptyset, X, \{1\}, \{2\}\}$ is not a topology because it is not closed under union.

3. Given any $X \neq \emptyset$, $\mathcal{T}_{\text{indiscrete}} = \{\emptyset, X\}$ and $\mathcal{T}_{\text{discrete}} = \mathcal{P}(X)$. $\mathcal{T}_{\text{discrete}}$ is the set of all open sets in the discrete metric, so it is induced by the discrete metric.

In general, if a topology on X is induced by a metric, then the topological space is called metrizable. If a topological space is metrizable, we already know how to do analysis. But most often topological spaces are not metrizable.

4. $(\mathbb{R}, \mathcal{T}_{\text{ray}})$ where $\mathcal{T}_{\text{ray}} = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$.

5. Given $X \neq \emptyset$, $\mathcal{T}_{\text{finite}} = \{A \subseteq X : A = \text{finite}\}$ is not generally a topology. For one thing, if X is infinite then X isn't in the topology! But even if we add $\{X\}$ to the topology, there's a problem: $\mathcal{T}_{\text{finite}}$ is not closed under infinite unions. However,

$$\mathcal{T}_{\text{cofinite}} = \{A \subseteq X : A^c = \text{finite}\} \cup \{\emptyset, X\}$$

is a topology.

6. Given $X \neq \emptyset$, $\mathcal{T}_{\text{cocountable}} = \{A \subseteq X : A^c = \text{countable}\} \cup \{\emptyset, X\}$ is a topology.

7. The particular point topology: Given $X \neq \emptyset$, pick $\alpha \in X$. $\mathcal{T}_\alpha = \{A \subseteq X : \alpha \in A\} \cup \{\emptyset\}$. If you replace α with a subset of X, this is also a topology.

3. Comparing Topologies

On $\mathbb{R}$, $\mathcal{T}_{\text{ray}} \subseteq \mathcal{T}_{\text{usual}}$. This means that $\mathcal{T}_{\text{usual}}$ is more refined than $\mathcal{T}_{\text{ray}}$, and $\mathcal{T}_{\text{ray}}$ is more coarse. We can think of a topology being more refined if it has finer distinctions between what it recognizes as open. Sometimes, we have two topologies where neither is a refinement of the other. For example, if we try to compare $\mathcal{T}_{\text{ray}}$ with $\mathcal{T}_7$, we will see that $(8, \infty) \in \mathcal{T}_{\text{ray}}$ but $(8, \infty) \notin \mathcal{T}_7$, and $\{7\} \in \mathcal{T}_7$ but $\{7\} \notin \mathcal{T}_{\text{ray}}$.

4. Basis of Topology

Recall that in $\mathbb{R}_{\text{usual}}$, $\mathcal{T}_{\text{usual}} = \{\mathcal{O} \in \mathbb{R} : \forall x \in \mathcal{O}, \exists \delta > 0 \text{ such that } (x - \delta, x + \delta) \subseteq \mathcal{O}\}$. Then the easier description is that $\mathcal{T}_{\text{usual}}$ is generated by open intervals, i.e. it is the collection of all the unions of open intervals.

But can we do this in general?

Given a set $X \neq \emptyset$, we want to find a collection $\mathcal{B} \subseteq \mathcal{P}(X)$ of particularly 'nice' open sets such that all possible unions of them forms a topology on X. What are some necessary conditions on $\mathcal{B}$ for this to happen? Well, for one thing, the topology must contain X as an element, so $\mathcal{B}$ must 'cover' X: it must be possible to represent X as a union of elements of $\mathcal{B}$. Equivalently, the union of all elements in $\mathcal{B}$ must equal X.

What other necessary conditions are there for the collection of all unions of elements of $\mathcal{B}$ to form a topology? We need finite intersections of elements of $\mathcal{B}$ to be in the topology, i.e. we need finite intersections of elements of $\mathcal{B}$ to be unions of elements of $\mathcal{B}$.

These two necessary conditions for $\mathcal{B}$ to generate a topology turn out to be sufficient as well! Here's a formal definition:

Definition (Basis)**.** Given $X \neq \emptyset$, we say $\mathcal{B} \subseteq \mathcal{P}(X)$ is a basis if and only if:

(i) $\bigcup_{S \in \mathcal{B}} S = X$

(ii) $J \cap K$ must be a union of elements in $\mathcal{B}$ for all $J, K \in \mathcal{B}$

Proposition 4.1. *If $\mathcal{B}$ is a a basis for $\mathcal{T}$, then $\mathcal{T}$ is generated by $\mathcal{B}$, i.e. $\mathcal{T} = \{\bigcup_{S \in A} S : A \subseteq \mathcal{B}\}$ is a topology on X.*

Proof. See course website. □

5. A ridiculous application

While a college student, Furstenberg discovered a ridiculous topological proof of the infinitude of prime numbers.

Theorem 5.1. *There are infinitely many prime numbers.*

Furstenberg's topological proof. Consider $\mathcal{B} := \{a\mathbb{Z} + b : a, b \in \mathbb{Z}\}$, i.e. all bi-infinite arithmetic progressions. This is a basis (check this!), hence it generates a topology on $\mathbb{Z}$. For any positive integer n, the set $n\mathbb{Z}$ of all multiples of n is open by definition. However, it is also closed:

$$(n\mathbb{Z})^c = (1 + n\mathbb{Z}) \cup (2 + n\mathbb{Z}) \cup ... \cup ((n-1) + n\mathbb{Z})$$

is a union of open sets, hence is open. If there were only finitely many primes p, then $\bigcup_{p=\text{prime}} p\mathbb{Z}$ would be closed. But $(\bigcup_p p\mathbb{Z})^c = \{\pm 1\}$ can't be open, because it's not a union of elements in the basis. □

6. Miscellaneous Tangents

Conjecture 6.1. *Suppose S is a finite collection of finite sets that's closed under union. Then $\exists$ a popular element, ie. $\exists$ x belonging to at least 50% of the elements of S.*