TOPOLOGY: LECTURE 7

Ben Shapiro and Robert Nielsen

Last Time

In order to iron out any confusions from the previous lecture, we began by once again defining the term basis.

Definition 1. Given $X \neq \emptyset$, a basis on X is $\mathcal{B} \subseteq \mathcal{P}(X)$ such that:

1. $\mathcal{B}$ covers X, that is $\bigcup_S \in \mathcal{B}$
2. $\forall J, K \in \mathcal{B}, J \cap K =$ the union of elements in $\mathcal{B}$

Then the topology generated by $\mathcal{B}$ is all possible unions of elements of $\mathcal{B}$ (including the empty union).

Just to clarify, the notion of a basis is that it comprises the "building blocks" of a topology. It allows us to take the union of nice open sets to get all of the elements in a topology. Think about it like unions of open intervals in $\mathbb{R}$.

Sorgenfrey Line

We then considered the set $\mathcal{B} = \{[\alpha, \beta) \mid \alpha, \beta \in \mathbb{R}\}$ and asked whether or not $\mathcal{B}$ is a basis on $\mathbb{R}$. We decided that $\mathcal{B}$ satisfied the first property found in the definition of basis. That is, $\mathcal{B}$ covers $\mathbb{R}$. Next, we showed that the intersection of two intervals in $\mathcal{B}$ is also in $\mathcal{B}$. Hence, we were able to say that $\mathcal{B}$ did in fact form a basis on $\mathbb{R}$. The topology generated by $\mathcal{B}$ is called the *Lower Limit Topology*, and is denoted $\mathcal{T}_{lower\ limit}$. Additionally, $\mathbb{R}$ with respect to $\mathcal{T}_{lower\ limit}$ is called the *Sorgenfrey Line*, and is denoted $\mathbb{R}_{Sorgenfrey}$.

In a brief aside, Levi brought up the following proposition:

Proposition 1. The topology generated by a basis $\mathcal{B}$ is the coarsest[1] topology containing $\mathcal{B}$.

Returning to the Sorgenfrey Line, Sam asked whether it was possible to for a countable basis for $\mathcal{T}_{lower\ limit}$. Leo said that this was a great question and that we will return to it later. The Sorgenfrey Line is an interesting example of a topological space because it has some expected properties and some very unexpected properties. Here are a few that we talked about:

Examples.

1. $\mathcal{T}_{lower\ limit}$ refines $\mathcal{T}_{usual}$
2. $\mathbb{R} = \mathcal{O}_1 \sqcup \mathcal{O}_2$ where $\mathcal{O}_1, \mathcal{O}_2$ are open, nonempty sets with respect to $\mathcal{T}_{lower\ limit}$. That is, $\mathbb{R}$ can be written as the disjoint union of two nonempty, open sets with respect to $\mathcal{T}_{lower\ limit}$.
3. $\forall S \in \mathcal{T}_{lower\ limit}$, S is clopen.
4. Consider $A = \{\frac{1}{n} \mid n \in \mathbb{Z}_{>0}\}$. Then, $\overline{A} = A \cup \{0\}$. This is an expected property.
5. Consider $C = \{\frac{-1}{n} \mid n \in \mathbb{Z}_{>0}\}$. Then, $\overline{C} = C$. This is an unexpected property, but we see later in this lecture why this is the case.

[1]The word "smallest" was originally used here, but after some discussion we decided that "coarsest" better articulated the notion that we were attempting to convey in this proposition.

Sub-basis

Definition 2. Given $X \neq \emptyset$. If the topology $\mathcal{T}$ is the smallest topology on X containing a set $\subseteq \mathcal{P}(X)$, then S is a sub-basis of $\mathcal{T}$. That is, given any set $S \subseteq \mathcal{P}(X)$, S generates a topology $\mathcal{T}$ on X, and S is called a sub-basis of $\mathcal{T}$.

Example. Given $X \neq \emptyset$, the topology generated by $\{\emptyset\}$ is the indiscrete topology $\mathcal{T}_{indiscrete} = \{\emptyset, X\}$

Proposition 2. Given $(X, \mathcal{T})$ with subbasis S, then $\mathcal{B} :=$ {all finite intersections of elements of S} is a basis on S generating $\mathcal{T}$. Notably, this includes empty intersections.

Sequences in Topological Spaces

We first looked back at sequences in metric spaces to calibrate ourselves before defining sequences and limits in topological spaces. We recalled that in a metric space (X, d), $(a_n) \to L$ iff $\forall \varepsilon > 0$, $d(a_n, L) < \varepsilon$ $\forall$sufficiently large n. In an attempt to hide the metric, we reformatted this double implication to read: $(a_n) \to L$ iff $\forall$open ball $\mathcal{B}$ around L, $a_n \in \mathcal{B}$ $\forall$large n.

Next, we conjectured about what $(a_n) \to L$ should mean in a topological space. Our guess was: $(a_n) \to L$ iff $\forall$open set $\mathcal{O}$ with $l \in \mathcal{O}$, $a_n \in \mathcal{O}$ $\forall$large n. Before going forth into topological spaces, we first checked that this proposition holds in metric spaces.

Proposition 3. In any metric space (X, d), $(a_n) \to L$ iff $\forall$open set $\mathcal{O}$ with $L \in \mathcal{O}$, $a_n \in \mathcal{O}$ $\forall$large n.

Proof. ($\longleftarrow$) Open balls are open sets, so the definition of convergence in a metric space holds. ($\longrightarrow$) Suppose $(a_n) \to L$. Pick any open set $\mathcal{O}$ with $L \in \mathcal{O}$. Then, L is an interior point of $\mathcal{O}$. So, there exists some open ball $\mathcal{B}$ around L with $\mathcal{B} \subseteq \mathcal{O}$. Since $(a_n) \to L$, $a_n \in \mathcal{B}$ $\forall$large n. Therefore, $a_n \in \mathcal{O}$ $\forall$large n. $\square$

With confidence since our conjecture held in metric spaces, we made the following definition.

Definition 3. In any topological space $(X, \mathcal{T})$, $(a_n) \to L$ iff $\forall$open set $\mathcal{O}$ with $L \in \mathcal{O}$, $a_n \in \mathcal{O}$ $\forall$large n.

We ended the lecture by looking at some example sequences and their limits in various topological spaces.

Example.

1. In $\mathbb{R}_{Sorgenfrey}$, $\lim_{n\to\infty} \frac{1}{n} = 0$
   Proof. Pick any open set $\mathcal{O} = [\alpha, \beta)$ with $0 \in \mathcal{O}$. That is, $\alpha \leq 0 < \beta$. For this sequence, $\frac{1}{n} \in (0, \beta)$ $\forall$large n. Thus, $\frac{1}{n} \in \mathcal{O}$ $\forall$large n. $\square$

2. In $\mathbb{R}_{Sorgenfrey}$, $\lim_{n\to\infty} \frac{-1}{n} 0$ because $[0, 1)$ is open but $\frac{-1}{n} \notin [0, 1) \forall n$

3. In $\mathbb{R}_{discrete}$, $\lim_{n\to\infty} \frac{-1}{n} \neq 0$ because $\{0\}$ is open with respect to this topology, but $\frac{1}{n} \notin \{0\}$ $\forall n$.

   More generally, in $(X, \mathcal{T}_{discrete})$, $(a_n) \to L$ iff $a_n = L$ $\forall$.

4. In $(\mathbb{R}, \mathcal{T}_7)$, $3, 7, 3, 7, 3, 7, \ldots \to 3$ because any open set containing 3 must contain 7 by the definition of the topology. By contrast, $0, 1, 0, 1, 0, 1, \ldots$ diverges.