TOPOLOGY: LECTURE 8

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Abstract. Lecture summary for Tuesday Oct. 4.

0. Recall

Last class, we discussed examples of convergence.

- In $\mathbb{R}_{sorgenfrey}$, $(\frac{1}{n}) \to 0$, however $(\frac{1}{-n}) \not\to 0$
- In $(X, \mathcal{T})$, (a_n) converges if and only if $a_n = L\ \forall$ large n, where $L \in \mathcal{T}$
- In the particular point topology on $\mathbb{R}$, with 7 as our point, we see:
  - The sequence $(7, 7, 7, 7, ...)$ can converge to anything, because all open sets in this topology will contain 7.
  - The sequence $(7, 3, 7, 3, 7, ...)$ will converge to 3 because sets in this topology that contain 3 will always contain 7, but sets that contain 7 need not contain 3.
  - The sequence $(1, 3, 7, 1, 3, 7, 1, 3, ...)$ does not converge, sets containing 1 don't have to contain 3, and vice versa.
  - The sequence $(3, 3, 3, ...)$ converges to 3.
- In $\mathbb{R}_{indiscrete}$, anything can converge to anything.
- In $\mathbb{R}_{ray}$, $(\frac{1}{n}) \to$ any negative number.

1. Asking the Real Questions: Why is Convergence so Messed Up?

We note that when $(a_n) \to L$, this conceptually means that no matter how far we zoom in on L, we will see the set $\{a_n\}$. This doesn't work nicely in every topological space. For example:

In $\mathcal{T}_{indiscrete}$, we can't distinguish any of our elements with our "open set microscope".

Oppositely, in $\mathcal{T}_{discrete}$, we are able to distinguish all of our elements because the singletons are open.

What in the world are we going to do about this?

Template by Leo Goldmakher.

2. Conditions on Toplogical Spaces

Since we see that $\mathcal{T}_{indiscrete}$ fails to distinguish between two arbitrary elements, which leads to undesirable properties, we want to impose some conditions on a topology such that we avoid these properties.

Definition. A topological space $(X, \mathcal{T})$ is called T_0 if $\forall\ \alpha, \beta \in X$ such that $\alpha \neq \beta,\ \exists \mathcal{O} \in \mathcal{T}$ that contains precisely one of α or β.

Example 1. The particular point topology, $(\mathbb{R}, \mathcal{T}_7)$, is T_0 because given $\alpha, \beta \in \mathbb{R}$, where $\alpha \neq 7$ and $\beta \neq 7$, $S = \{\alpha, 7\}$ is open and $\beta \notin S$. Further, given $\alpha, 7 \in \mathbb{R}$, $\{7\}$ is open, but $\alpha \notin \{7\}$.

Example 2. The indiscrete topology, $(X, \mathcal{T}_{indiscrete})$ is not T_0 because for any $\alpha, \beta \in X$ the only open set containing α or β is the entire space X, so there is no open set that contains one but not the other.

Example 3. The leftwards ray topology on $\mathbb{N}$ is T_0 because given $m, n \in \mathbb{N}$, where $m < n$ there exists an open set containing m that does not contain n, namely, $\{0, 1, ..., m-1, m\}$.

Even though T_0 eliminates the indiscrete toplogy, it still allows for topologies in which a constant sequence (α) can converge to something other than α, so it is not particularly useful. For this reason, we we'd like to have some slightly stronger conditions.

Definition. Given a toplogical space $(X, \mathcal{T})$, we call it T_1 if for some $\alpha, \beta \in X$ such that $\alpha \neq \beta$, $\exists\ \mathcal{O}_\alpha, \mathcal{O}_\beta \subseteq \mathcal{T}$ such that $\alpha \in \mathcal{O}_\alpha \setminus \mathcal{O}_\beta$ and $\beta \in \mathcal{O}_\beta \setminus \mathcal{O}_\alpha$.

Example 4. The particular point topology, $(\mathbb{R}, \mathcal{T}_7)$ is not T_1 because given $\alpha \neq 7 \in \mathbb{R}$, $\forall \mathcal{O} \in \mathcal{T},\ \alpha \in \mathcal{O} \implies 7 \in \mathcal{O}$

Example 5. $\mathbb{R}_{ray}$ is not T_1 because given $x > y \in \mathbb{R}$, every open set containing y also contains x.

Proposition 2.1. *If $(X, \mathcal{T})$ is T_1, then every constant sequence (α) converges only to α.*

Proof. Suppose the sequence $(\alpha, \alpha, \alpha, ...) \to \beta \neq \alpha$ generates a contradiction because there must be some open set containing β that does not contain α. □

An issue that remains, though, is that, in a T_1 space, we can zoom in separately on α and β, but the zooms might have overlap.

Example 6. In $\mathbb{R}_{cofinite}$, the sequence $(\frac{1}{n}) \to 0$ because any open set containing 0 can only exclude finitely many elements of the sequence. But, by the same logic, this sequence converges to any element of $\mathbb{R}$.

So, even though T_1 spaces have some nice properties, a sequence can still converge to multiple limits. To avoid this problem, we impose further restrictions on topological spaces:

Definition. Given a topological space $(X, \mathcal{T}), \forall \alpha \neq \beta \in X, \exists$ open sets $\mathcal{O}_\alpha, \mathcal{O}_\beta \in \mathcal{T}$ such that $\alpha \in \mathcal{O}_\alpha$ and $\beta \in \mathcal{O}_\beta$, with $\mathcal{O}_\alpha \cap \mathcal{O}_\beta = \varnothing$. We call a topological space that satisfies these conditions T_2

Proposition 2.2. *If $(X, \mathcal{T})$ is a T_2 topology, then (a_n) converges to at most one limit.*

Proof. Suppose $(a_n) \to L$ and $(a_n) \to \alpha$ with $\alpha \neq L$ in a T_2 topology. Then, $\exists$ disjoint opens $\mathcal{O}_\alpha, \mathcal{O}_L$ with $\alpha \in \mathcal{O}_\alpha$ and $L \in \mathcal{O}_L$. $\forall$ large n, $(a_n) \in \mathcal{O}_\alpha$ and $(a_n) \in \mathcal{O}_L$. However, from our T_2 condition, $\mathcal{O}_\alpha \cap \mathcal{O}_\beta = \varnothing$, a contradiction. □

These spaces are sometimes known by other names that may be easier to remember:

- T_0 is occasionally called Kolmogorov space.
- T_1 is occasionally known as a Fréchet space.
- T_2 is usually called Hausdorff space.