LECTURE SUMMARY

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0. Note on pset:

Your proof that $\mathbb{R}_l$, the Sorgenfrey line, does not admit a countable basis should not imply that the normal topology on $\mathbb{R}$ does not admit a countable basis. It does; for instance, $\mathcal{B} = (a, b) : a, b \in \mathbb{Q}$ is a countable basis since every open interval contains a sub-interval with rational endpoints.

1. Last time:

If you want to tell points apart by open sets, you can impose the T_0 separation axiom: given two points in our topological space, there is an open set that only contains either one of them.

This axiom is basically the least structure needed to be able to tell all points apart by open sets. Because it is such a weak condition, it does not imply many nice properties; for instance, in $(\mathbb{R}, \tau_7)$, the particular point topology at 7, which is T_0, the constant sequence $7, 7, 7, 7, 7 \to \pi$.

If you want a little better structure, you can impose the stronger axiom T_1: given two points in this space, there is an open set containing the first and not the second (and one containing the second and not the first).

This is a much nicer condition; for instance, constant sequences only converge to what they should in this space, as $(a, a, a, \dots) \not\to b$ since by T_1 there is an open set containing b but not a. Furthermore,

Proposition 1.1. *A topological space is T_1 iff singletons are closed in it.*

($\Rightarrow$) if $x \in X$, for all $y \in X$ such that $y \neq x$ there is an open set U_y containing y and not x. Then $\bigcup_{y \neq x} U_y = X \setminus \{x\}$ is a union of opens, and thus open, so $\{x\}$ is closed.

($\Leftarrow$) if all singletons are closed, for any points a, b the sets $X \setminus \{a\}$ and $X \setminus \{b\}$ are open, and each contains one but not the other, demonstrating that our space is T_1. $\square$

Date: October 12, 2022.

Template by Leo Goldmakher.

However, we can still have some weird things happening for convergence in T_1, such as in $\mathbb{R}_{\text{cofinite}}$, $\frac{1}{n} \to \pi, \pi^2, 7, \ldots$. This is because although any two points can be distinguished by open sets, those open sets may have to overlap, so you could have points that are arbitrarily close to multiple points.

To fix this, we can impose the even stronger axiom T_2, the Hausdorff condition: these spaces are the nicest, and the condition implies many nice and intuitive properties. For instance,

Proposition 1.2. *If (X, τ) is Hausdorff, then every convergent sequence in it has a unique limit.*

Let $(x_0, x_1, x_2, \ldots) \to x$ and assume for contradiction that $(x_0, x_1, x_2, \ldots) \to y$ also. Now by the T_2 axiom take open sets $U_x \ni x$ and $U_y \ni y$ such that $U_x \cap U_y = \varnothing$. Then by definition of limits, for sufficiently high n, $x_n \in U_x$ and $x_n \in U_y$, but $U_x \cap U_y = \varnothing$, a contradiction. $\square$

(Note that the converse does not hold! For instance, the cocountable topology on $\mathbb{R}$ has unique sequential limits but is not Hausdorff.)

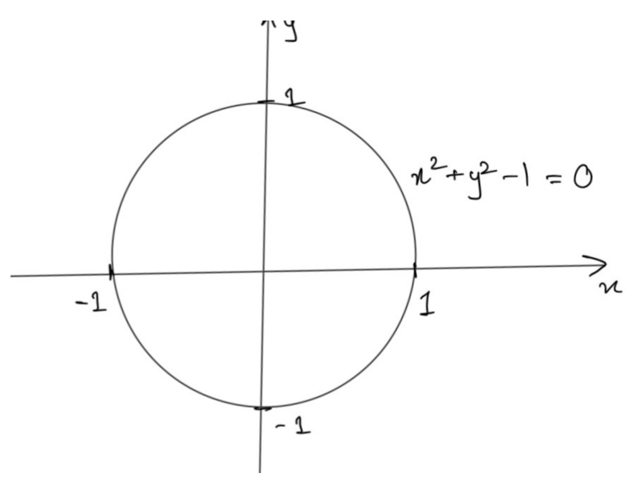
2. An important example:

Consider the algebraic numbers, i.e., roots of polynomials with integer coefficient; algebraically, these numbers are less "nice" than the rationals, but more "nice" than arbitrary numbers in $\mathbb{C}$. You can describe an algebraic number, for example, as the largest root to $(x^5 - x - 1)$.

The algebraic numbers are the solutions to single-variable polynomials, but the solutions to multivariable polynomials are also very well-behaved, eg: the solutions to

$$x^2 + y^2 - 1 = 0$$

are easily described as the points lying on the unit circle.



Let $\overline{\mathbb{Q}} = \{\text{all algebraic numbers}\}$. This is a field, i.e., a set equipped with a commutative addition, a commutative multiplication, and appropriate subtraction and division to act as those operations' inverses, with some compatibility conditions.

We want the roots of a polynomial $p(x_1, \ldots, x_n) \in \mathbb{R}[x_1, x_2, ..., x_n]$ (the ring of polynomials in n variables with coefficients in $\mathbb{R}$) to form a closed set.

This means that we want the set $\text{supp}(p) = \{(a_1, \ldots, a_n) \mid p(a_1, \ldots, a_n) \neq 0\}$ to be open.

Proposition 2.1. $\mathcal{B} := \{\text{supp}(p) \mid p \in \mathbb{R}[x_1, x_2, ..., x_n]\}$ *is a basis.*

Let $U_1 = \text{supp}(p), U_2 = \text{supp}(q) \in \mathcal{B}$; then

$$\begin{aligned} U_1 \cap U_2 &= \text{supp}(p) \cap \text{supp}(q) \\ &= \{(a_1, \ldots, a_n) \mid p(a_1, \ldots, a_n) \neq 0\} \cap \{(a_1, \ldots, a_n) \mid q(a_1, \ldots, a_n) \neq 0\} \\ &= \{(a_1, \ldots, a_n) \mid p(a_1, \ldots, a_n) \neq 0 \wedge q(a_1, \ldots, a_n) \neq 0\} \\ &= \{(a_1, \ldots, a_n) \mid p(a_1, \ldots, a_n) \cdot q(a_1, \ldots, a_n) \neq 0\} \\ &= \{(a_1, \ldots, a_n) \mid (p \cdot q)(a_1, \ldots, a_n) \neq 0\} \\ &= \text{supp}(p \cdot q) \in \mathcal{B} \end{aligned}$$

Moreover, $\text{supp}(1) = \mathbb{R}^n$, so $\mathcal{B}$ covers the whole space and intersections can be rewritten as unions, so $\mathcal{B}$ is a basis. □

This generates the "Zariski topology", a T_1 space (when coefficients are in $\mathbb{R}$), but badly not T_2, since any two non-empty sets intersect in this topology.

3. Analysis:

Given (X, τ) and $A \subseteq X$, what is $\text{int}A$?

Definition (Max)**.** $\text{int}A$ is the largest open subset of A, i.e., $\bigcup_{\mathcal{O} \in \tau, \mathcal{O} \subseteq A} \mathcal{O}$.

Note that this definition is dual to closure:

$$\overline{A} = \bigcap_{X \setminus C \in \tau, A \subseteq C} C$$

Definition (Elias)**.** $x \in \text{int}A$ iff $\exists \mathcal{O} \in \tau$, such that $x \in \mathcal{O} \subseteq A$.

These two definitions are evidently equivalent.

Similarly, the boundary of A can be defined as:

Definition (Sam)**.** $\partial A = \overline{A} \setminus \text{int}A$

Definition (Lizzie)**.** $\partial A = \overline{A^C} \setminus \text{int}A^C$

These are equivalent as $\overline{A^C} = (\text{int}A)^C$ and $\text{int}A^C = \overline{A}^C$, and if $A \subseteq B$ then $B \setminus A = A^C \setminus B^C$.

Definition. $x \in \partial A$ iff $\forall \mathcal{O}$, such that $x \in \mathcal{O}$, $\mathcal{O} \cap A \neq \emptyset$ and $\mathcal{O} \cap A^C \neq \emptyset$.

This too is equivalent as if there was an open neighborhood of x contained in A, then x would belong to the interior, and if there was an open neighborhood of x contained in A^C, then that neighborhood's complement would contain A, so x would not be in the closure.

Proposition 3.1. $X = \text{int}A \sqcup \partial A \sqcup \text{int}A^C$

This is basically just unwinding definitions; it just means that $(\text{int}A^C)^C \setminus \text{int}A = \partial A$, but $(\text{int}A^C)^C$ is just $\overline{A}$, and $\overline{A} \setminus \text{int}A$ is just our definition of ∂A.

4. Continuity

Recall that in metric spaces $(X, d), (Y, e), f : X \to Y$ is continuous at $\alpha \in X$ iff $\forall \epsilon > 0$ (perturbations in Y), $\exists \delta > 0$ (perturbations in X) such that

$$d(x, \alpha) < \delta \implies e(f(x), f(\alpha)) < \epsilon$$

or equivalently,

iff $\forall \epsilon > 0$, $\exists \delta > 0$:

$$x \in B_\delta(\alpha) \implies f(x) \in B_\epsilon(f(\alpha))$$

Relaxing our conditions to accommodate a general topological space with a basis, we get

f is continuous at α if for any basic opens B around $f(\alpha)$ there exists a basic open B' around α such that

$$x \in B' \implies f(x) \in B$$

or alternatively written:

$$B' \subseteq f^{-1}(B)$$

This motivates our definition of continuity in topological spaces:

Definition. Given topological spaces (X, τ) and (Y, ρ), and a function $f : X \to Y$,

f is continuous at $\alpha \in X$ if for all $\mathcal{O}_1 \in \rho$ such that $f(\alpha) \in \mathcal{O}_1$, there exists an open set $\mathcal{O}_2 \in \tau$ with $\alpha \in \mathcal{O}_2$ and $\mathcal{O}_2 \subseteq f^{-1}(\mathcal{O}_1)$.