Topology : Lecture 10

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Continuity

From Problem set 5,

Definition 1. $f : X \to Y$ is continuous at a point $\alpha \in X$ if and only if for any open set $\mathcal{O}$ in Y containing $f(\alpha)$ we have that α is in the interior of $f^{-1}(\mathcal{O})$ (which is the preimage of $\mathcal{O}$).

Proposition 1. *$f : X \to Y$ is continuous if and only if $f^{-1}(\mathcal{O})$ is open $\forall$ open $\mathcal{O} \subseteq Y$.*

Example 1. Question: do we have a "converse" definition of continuity? Namely, is it true that f is continuous $\iff$ $f(0)$ is open $\forall$ open $\mathcal{O} \subseteq X$?

No!

For example, take $f : \mathbb{R} \to \mathbb{R}$ under the usual topology given by $f(x) = 0$. Then $f((0,1)) = \{0\}$, but $\{0\}$ is not open under the usual topology.

Example 2. Consider $f : \mathbb{R}_{\text{discrete}} \to \mathbb{R}_{\text{usual}}$ given by

$$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 2022 & x = 0 \end{cases}$$

Then f is continuous, because given any open set $\mathcal{O} \in \mathbb{R}_{\text{usual}}$, the preimage of $\mathcal{O}$ will be some set $f^{-1}(\mathcal{O})$ in $\mathbb{R}$ (it doesn't matter what kind of set). But in the discrete topology, every set is open, so $f^{-1}(\mathcal{O})$ is open, so f is continuous.

Example 3. Consider $f : \mathbb{R}_{\text{indiscrete}} \to \mathbb{R}_{\text{usual}}$ given by $f(x) = x$. This is **not** continuous because $f^{-1}((0,1)) = (0,1)$, which is not open. The problem is that almost no sets are open in $\mathbb{R}_{\text{indiscrete}}$.

Example 4. Consider $f : \mathbb{R}_{\text{indiscrete}} \to \mathbb{R}_{\text{usual}}$ given by $f(x) = 0$. Then f is continuous. Too see this, take some open $\mathcal{O} \subseteq \mathbb{R}_{\text{usual}}$. If $0 \in \mathcal{O}$, then $f^{-1}(\mathcal{O}) = \mathbb{R}$, which is open. If $0 \notin \mathcal{O}$, then $f^{-1}(\mathcal{O}) = \emptyset$, which is also open.

Example 5. Consider $f : \mathbb{R}_{\text{Sorgenfrey}} \to \mathbb{R}_{\text{indiscrete}}$ given by

$$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 2022 & x = 0 \end{cases}.$$

Then f is continuous, because we only have two open sets, $\mathbb{R}$ and $\emptyset$, and we immediately have that $f^{-1}(\mathbb{R}) = \mathbb{R}$ and $f^{-1}(\emptyset) = \emptyset$.

In fact, any function which maps to a set under the indiscrete topology will always be continuous.

Example 6. Consider some general $f : (X, \mathcal{T}) \to (X, \mathcal{S})$ given by $f(x) = x$. This is continuous if and only if $\mathcal{T}$ is a refinement of $\mathcal{S}$, which means that every open set in $\mathcal{S}$ is also open in $\mathcal{T}$.

General Philosophy Continuity measure not just how nice our function f is, but also how refined or coarse the topologies are.

Take some $f : X \to Y$. If the topology on X is very refined, meaning it has a lot of open sets, then it is "easy" to be continuous. Likewise, if Y is very coarse, meaning it has very few open sets, then it is "easy" to be continuous.

Example 7. Consider $\pi : \mathbb{R}^2_{\text{usual}} \to \mathbb{R}_{\text{usual}}$ given by $\pi((x, y)) = x$ (here (x, y) is a point, not an interval). This is normally called a projection mapping. π is continuous. To see this, the preimage of any open interval (a, b) is open, because it is the "strip"

$$\{(x, y) : x \in (a, b) \text{ and } y \in \mathbb{R}\}$$

which is open in $\mathbb{R}^2_{\text{usual}}$. But we are not done, because there are other open sets in $\mathbb{R}_{\text{usual}}$, which are the union of open intervals. But in fact, if we take some open set $\mathcal{O} \subseteq \mathbb{R}_{\text{usual}}$, then $\mathcal{O}$ is the union of open intervals, so $\pi^{-1}(\mathcal{O})$ will also be a union of open sets, so we are done.

The above example shows a more general strategy. If we want to check if a function is open, we need to check that the preimage of every open set is open. However, this is equivalent to showing that the preimage of every basis element is open.

Connectedness

Intuitively, $\mathbb{R}$ is a connected set, while the union $[0, 1] \cup [2, 3]$ is disconnected. But what does it mean for a general topological space $(X, \mathcal{T})$ to be connected? Instead, lets think about what it means for $(X, \mathcal{T})$ to be disconnected.

Definition 2. $(X, \mathcal{T})$ is disconnected if and only if there exist open sets $A, B \neq \emptyset$, such that $X = A \sqcup B$.

Example 8. Consider $X = [0, 1] \cup [2, 3]$ under the usual topology. Then $[0, 1]$ and $[2, 3]$ are open (and in fact clopen), so X is disconnected because $X = [0, 1] \sqcup [2, 3]$.

There is another way to think about disconnectedness in terms of clopen sets.

Proposition 2. *$(X, \mathcal{T})$ is disconnected if and only if $\exists$ clopen $\emptyset \neq A \not\subseteq X$.*

In other words, $(X, \mathcal{T})$ is connected if and only if the only clopen sets are the empty set and the entire space.

There is yet another way to think about disconnectedness.

Proposition 3. *$(X, \mathcal{T})$ is disconnected if and only if $X = A \sqcup B$ such that A, B are nonempty and $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$.*

The proofs of these two equivalent definitions of disconnectedness will be left as an exercise.

Now, we haven't actually defined connectedness, but this is easy enough.

Definition 3. A space $(X, \mathcal{T})$ is connected if and only if it is not disconnected.

Example 9. $\mathbb{R}_{\text{usual}}$ is connected, but proving this is somewhat involved (we will do it next time).

Example 10. $\mathbb{R}_{\text{Sorgenfrey}}$ is disconnected. The basis for the topology are the half open intervals $[a, b)$. We have shown previously that $(-\infty, 0)$ is open, so $\mathbb{R} = (-\infty, 0) \sqcup [0, \infty)$ is a disjoint union of open sets.

Example 11. $\mathbb{R}_{\text{ray}}$ is connected, because any two disjoint open sets (which are non empty) intersect.