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## Williams College Department of Mathematics and Statistics

# MATH 374 : TOPOLOGY

#### **Oral Midterm Exam**

### Monday & Tuesday, November 4th & 5th

### INSTRUCTIONS

The midterm exam will consist of two questions (see next page), to be discussed orally. The duration of the exam is intended to be 30 minutes total: 10 minutes individually, followed by 20 minutes discussing with me. The exam will take place in my office, Wachenheim 337. Please don't be late to your scheduled appointment, as my schedule will be relatively unforgiving during the exam period and I won't be able to extend your time. You will have access to a blackboard, chalk, and blank paper; you may bring your own writing utensils, *but no other aids are permitted*—no notes, books, etc.

When you first arrive, we will select at random the two spaces from Question B; you will then have 10 minutes to work on that question on your own prior to starting the oral portion of the exam. As with the rest of the exam, you may not use any notes or other resources. (You may, however, bring writing utensils; I'll provide blank paper.) After taking 10 minutes on your own, the oral portion of the exam will begin. We'll start with Question A, which will be asked of every student. Then we'll turn to Question B; please note that you may *not* refer to your scratchwork when presenting your thoughts on Question B.

When explaining something during your exam, I ask that you present a high-level overview: no more than a few sentences, with as few technical details as possible. This will usually suffice, but I reserve the right to ask for more detail. This includes asking you to define any (mathematical) word that you use in your explanation.

Often, it is during an exam that you realize for the first time that you don't properly understand something. This is not only natural, it is totally OK; I will give you as many hints as you need to get back on track. Although part of the exam is to see how far you can go on your own, the more valuable aspect of an oral exam is that it's a chance for some individualized learning.

You may use any resources you like while preparing for the exam (both books and online), but the only humans you can discuss the exam with are me, other students currently enrolled in the course, and the TAs for the course. (In particular, no posting questions on online forums! AI resources are allowed, but I don't recommend it.) Moreover, once you have taken your exam, I request that you not discuss any aspect of it with other students until our class meeting the following Thursday, November 7th.

Best of luck, and please don't hesitate to reach out with any questions!

Leo

### MATH 374 : Midterm Exam

**Question A: Counterexample.** Every countable topological space we've encountered is either connected or Hausdorff but not both, and it seems plausible that this dichotomy holds for *any* countable space. Remarkably, this conjecture is false! Our goal is to construct a countable space that is both connected and Hausdorff.

Consider  $\mathbb{Q} \times \mathbb{Q}_{\geq 0}$  with respect to the topology  $\mathcal{T}_{sf}$ , defined as follows. (This is called *Bing's Sticky Foot Space*, for reasons you'll understand after playing around with it for a bit.) Given  $\mathbf{p} := (p_1, p_2) \in \mathbb{Q} \times \mathbb{Q}_{\geq 0}$ , set

$$\ell(\mathbf{p}) := \left(p_1 - \frac{p_2}{\sqrt{3}}, 0\right) \qquad r(\mathbf{p}) := \left(p_1 + \frac{p_2}{\sqrt{3}}, 0\right)$$

This has a geometric interpretation: the three points  $\mathbf{p}$ ,  $\ell(\mathbf{p})$ , and  $r(\mathbf{p})$  are the vertices of a (possibly degenerate) equilateral triangle whose base lies on the horizontal axis. Now for any  $\epsilon > 0$ , define

$$B_{\epsilon}(\mathbf{p}) := \{\mathbf{p}\} \cup \Big\{ \mathbf{x} \in \mathbb{Q} \times \{0\} : |\mathbf{x} - \ell(\mathbf{p})| < \epsilon \text{ or } |\mathbf{x} - r(\mathbf{p})| < \epsilon \Big\}.$$

(I highly recommend drawing a picture.) The collection of all such sets, i.e.

 $\mathcal{B} := \{ B_{\epsilon}(\mathbf{p}) : \mathbf{p} \in \mathbb{Q} \times \mathbb{Q}_{>0} \text{ and } \epsilon > 0 \},\$ 

is a basis on  $\mathbb{Q} \times \mathbb{Q}_{\geq 0}$ , hence generates a topology on  $\mathbb{Q} \times \mathbb{Q}_{\geq 0}$ ; we'll call this topology  $\mathcal{T}_{sf}$ .

Prove that  $\mathcal{B}$  is a basis on  $\mathbb{Q} \times \mathbb{Q}_{\geq 0}$ , and that  $\mathbb{Q} \times \mathbb{Q}_{\geq 0}$  with respect to  $\mathcal{T}_{sf}$  is connected and Hausdorff.

**Question B: Examples.** Two topological spaces from the list below will be selected at random via coin flips. If not otherwise specified, assume the usual / subspace topology.

<b>L.0</b> $\mathbb{R}_7$ (the particular point topology)	<b>L.8</b> $(0,1) \cup [2,3]$ (viewed as a subspace of $\mathbb{R}$ )
L.1 $\mathbb{R}_{cofinite}$	$\mathbf{L.9} \ \mathbb{Z}_{\mathrm{Furstenberg}}$
L.2 $\mathbb{R}_{\geq 0}$	$\mathbf{L.10} \ \mathbb{Q}_{\mathrm{usual}}$
L.3 $\mathbb{R}_{sorgenfrey}$	$\textbf{L.11} \ \times \ (\text{cross, viewed as a subspace of } \mathbb{R}^2)$
L.4 $\mathbb{R}_{ray}$	${\bf L.12}$ $\circ$ (circle, viewed as a subspace of $\mathbb{R}^2)$
$\mathbf{L.5} \ \mathbb{R}_{\mathrm{discrete}}$	${\bf L.13}$ The closure of a non-trivial open ball in $\mathbb{R}^2$
<b>L.6</b> $\mathbb{R}_{\text{indiscrete}}$	$\mathbf{L.14} \ \mathbb{R}^2 \setminus \{0\} \ (\text{here } 0 \text{ denotes the origin})$
<b>L.7</b> [0,1)	$\mathbf{L.15}~\mathbb{R}^2$ with respect to the British Rail metric

For each of your two spaces, answer the questions below. You should be able to justify your answers with proof sketches, but keep details to a minimum. If **L.9** and **L.10** are selected, I'll replace **L.10** by another space.

- (i) Describe the topology of the space. You may do this by either giving a direct description of all the open sets, or by describing a basis.
- (ii) Is the space  $T_0$ ? Is it  $T_1$ ? Is it Hausdorff?
- (iii) Is the space connected or disconnected?
- (iv) I'll give you a sequence; does it converge or diverge?
- (v) I'll give you a function from one space to the other; is it continuous?
- (vi) Are the two spaces homeomorphic?