Instructor: Leo Goldmakher

Williams College
Department of Mathematics and Statistics

MATH 374 : TOPOLOGY

Problem Set 1 – due Friday, September 16th

INSTRUCTIONS:

Please submit this assignment via Glow (to be published soon!) by 4pm on Friday. Your solution to problem **1.1** should be in LaTeX (ideally, the entire problem set, but one problem will suffice for now). If you aren't comfortable with LaTeX, or don't know what it is, don't worry—just reach out to me and we can figure it out.

This problem set will be graded for effort (as opposed to correctness), using a ✓+ / ✓/ ✓– scale. Have fun with it!

1.0 Watch this video about the game Hex. Then play at least ten rounds of the game! (Here's a nice website that allows you to play against either another human or an AI.)

[By default, the "swap rule" box is checked. I suggest unchecking it for at least your first few rounds.]

1.1 In the video linked above, a proof is presented that Hex always has a winner. Write down a clear exposition of this proof. I encourage you to change the order / terminology / notation of the proof: the goal is to write a proof that is crystal clear and can be followed without access to the video. [*As we'll see later in the course, this theorem has topological implications.*]

1.2 This problem is an introduction to *projective geometry*, a beautiful area of mathematics that emerged from developments in art during the Renaissance. Fix a point O (this stands for origin, but we're going to be working without reference to coordinates). We say two points T and T' are *in perspective* iff they lie on the same line through O. (If you imagine a light source at O, two points are in perspective iff one of them lies in the other's shadow.)

Suppose ABC and $A'B'C'$ are two generic triangles in the plane that are in perspective, i.e. corresponding vertices are in perspective. (By *generic* I mean the triangles are disjoint and that none of the edges are parallel.) Let

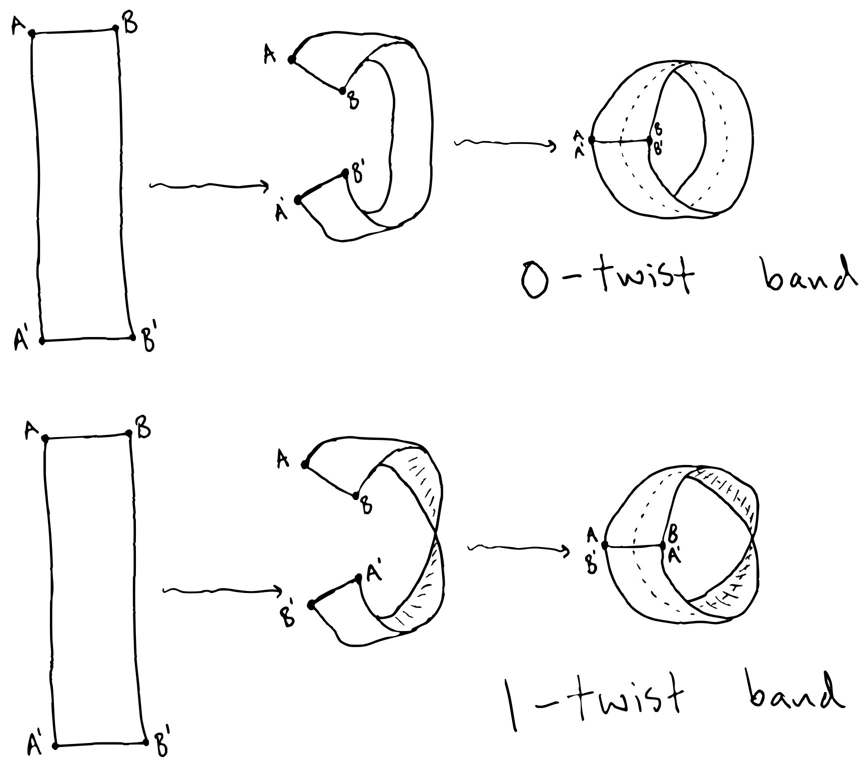
$$X := \overleftrightarrow{AB} \cap \overleftrightarrow{A'B'} \quad , \quad Y := \overleftrightarrow{AC} \cap \overleftrightarrow{A'C'} \quad , \quad Z := \overleftrightarrow{BC} \cap \overleftrightarrow{B'C'}$$

where $\overleftrightarrow{ST}$ denotes the unique line passing through the points S and T. Click here for an interactive illustration (interactive, in that you can drag any of the points around and the diagram will adjust appropriately).

Give a short explanation (no need for a formal proof) of why X, Y, Z must be collinear. [*Hint: Stare at an illustration and try to imagine the two triangles are generically positioned in* $\mathbb{R}^3$ *rather than both lying in the same plane.*]

1.3 Using a rectangular strip of paper, one can form a bracelet by bending the strip and gluing the edges (without twisting the strip along the way); I'll call this a *0-twist band*. If instead we introduce one twist before gluing the edges, we obtain a *1-twist band*, popularly known as the Möbius strip. (See below for an attempt at an illustration.) Similarly one can form an n*-twist band* by introducing n twists prior to gluing the edges together.

If you cut along the line halfway across the 0-twist band (indicated by the dotted line in the illustration below), the band falls apart into two identical 0-twist bands. If instead you cut along the line 1/3 of the way across the 0-twist band, the band falls apart into two 0-twist bands, one of which is twice as thick as the other.



(a) What would happen if you cut along the line halfway across the 1-twist band? Describe in words what you think should happen. You may use drawings to help you, but **do not physically construct this yet**. (Also, as always with problem sets, please don't do any online searches.)

(b) Having completed part (a), construct a physical model of a 1-twist band and cut it along the halfway line. Was your prediction in part (a) accurate? If not, can you now explain (in words) why you get what you get? *Please leave part (a) as you wrote it, whether or not your prediction was correct*; keep in mind that your problem set will be graded based only on effort, not on correctness.

(c) What would happen if you cut along the line 1/3 of the way across the 1-twist band? Describe in words what you think should happen. You may use drawings to help you, but **do not physically construct this yet**.

(d) Having completed part (c), construct a physical model of a 1-twist band and cut it along the 1/3 line. Was your prediction in part (c) accurate? If not, can you now explain (in words) why you get what you get? *Please leave part (c) as you wrote it, whether or not your prediction was correct.*