Instructor: Leo Goldmakher

Williams College Department of Mathematics and Statistics

MATH 374 : TOPOLOGY

Problem Set 2 - due Thursday, September 19th

INSTRUCTIONS:

You should aim to submit this assignment to me in person at the start of Thursday's class; if you cannot make it to class, email me by 11am on Thursday and we can discuss alternative ways to submit your assignment. Late assignments can be left in the mailbox outside my office until 4pm on Friday (incurring a small penalty, as described in the course syllabus). Assignments will not be accepted after 4pm on Friday.

2.1 For each of the following metrics on \mathbb{R}^2 , draw a picture the open ball $\mathcal{B}_3((2,0))$. No proofs necessary.

- (a) The chessboard metric $d(x, y) = \max\{|x_1 y_1|, |x_2 y_2|\}$.
- (b) The British Rail metric

$$d(x,y) := \begin{cases} |x| + |y| & \text{if } x \neq y \\ 0 & \text{otherwise} \end{cases}$$

(Here |x| denotes the Euclidean distance from x to the origin.)

(c) The discrete metric $d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$

- **2.2** Suppose (X, d) is a metric space and $\mathcal{A} \subseteq X$. We say $p \in X$ is an *interior point* of \mathcal{A} iff $\exists r > 0$ such that $\mathcal{B}_r(p) \subseteq \mathcal{A}$, and that $p \in X$ is a *limit point* of \mathcal{A} iff there exists a sequence (a_n) of points in $\mathcal{A} \setminus \{p\}$ such that $\lim_{n \to \infty} a_n = p$. (As always, $\mathcal{B}_r(p)$ denotes the ball of radius r around p.)
 - (a) Prove that \mathcal{A} is open iff every point of \mathcal{A} is an interior point of \mathcal{A} . (In class we defined: \mathcal{A} is open iff $\partial \mathcal{A} \cap \mathcal{A} = \emptyset$.)
 - (b) Prove that \mathcal{A} is closed iff every limit point of \mathcal{A} is in \mathcal{A} . (In class we defined: \mathcal{A} is closed iff $\partial \mathcal{A} \subseteq \mathcal{A}$.)
- **2.3** Suppose (X, d) is a metric space. Prove that $\mathcal{B}_r(p)$ is open for any $p \in X$ and any r > 0.
- 2.4 Decide (with proof or counterexample) whether each of the following is a metric space.
 - (a) $\mathbb{R}^{\infty} := \{(a_n) : (a_n) \text{ is a sequence of real numbers}\}, \text{ with respect to } d(x, y) := \max\{|x_n y_n|\}.$
 - (b) $\mathcal{F} := \{A \subseteq \mathbb{Z} : A \text{ is finite and nonempty}\}, \text{ with respect to } d(X,Y) := \log \frac{|X-Y|}{\sqrt{|X|}\sqrt{|Y|}}.$ Here |S| denotes the size of S and $X Y := \{x y : x \in X, y \in Y\}.$
- **2.5** Exploring metrics on \mathbb{R}^2 .
 - (a) Prove that the Euclidean metric on \mathbb{R}^2 is, in fact, a metric.
 - (b) Suppose \mathcal{O} is a subset of \mathbb{R}^2 that's open with respect to the Euclidean metric. Must it also be open with respect to the taxicab metric?

(c) The Euclidean and taxicab metrics on \mathbb{R}^2 both have the form

$$d_p(x,y) := \left(|x_1 - y_1|^p + |x_2 - y_2|^p\right)^{1/p}$$

 $(d_1 \text{ is the taxicab metric, } d_2 \text{ is the Euclidean metric})$. It turns out that d_p is a metric for every real number $p \geq 1$. (Don't worry about proving it here, although it is a fun challenge to think about when you have some spare time.) Can you describe any of the other metrics on \mathbb{R}^2 that we've encountered (chessboard, British Rail, and discrete) in terms of d_p ? No formal proofs necessary, but give a bit of justification for your answer.

- **2.6** Given a metric space (X, d) where X has at least 3 elements. Prove that there exists a metric on X that's not a scalar multiple of d or of the discrete metric.
- **2.7** Given (X, d) a metric space and $\mathcal{A} \subseteq X$. Prove that \mathcal{A} is closed iff \mathcal{A}^c is open.
- **2.8** In class we saw an example of a set that was open in \mathbb{R} , but neither open nor closed when viewed as sitting in \mathbb{R}^2 .
 - (a) If $A \subseteq \mathbb{R}$ (with respect to the Euclidean metric) is open, is it possible for $A \times \{1\}$ to be open in \mathbb{R}^2 (with respect to the Euclidean metric)? Either prove that it's never possible, or present an example where it is possible.
 - (b) If $A \subseteq \mathbb{R}$ (with respect to the Euclidean metric) is closed, is $A \times \{1\}$ closed in \mathbb{R}^2 (with respect to the Euclidean metric)? Either prove that this is always the case, never the case, or that it is sometimes the case and sometimes not by giving explicit examples.
- **2.9** Prove that a point cannot be simultaneously in the interior of A and on the boundary of A. (More generally, prove that A doesn't contain its boundary iff A consists of interior points.) Why doesn't this contradict our bizarre example from Lecture 3, in which we saw that [0,3) is open in $\mathbb{R}_{\geq 0}$ with respect to the Euclidean metric?
- **2.x** (*Optional challenge problem—won't be graded*) Let $M_{n \times n}$ denote the space of all $n \times n$ matrices with real entries. Prove that $d(x, y) := \operatorname{rank}(x y)$ is a metric on $M_{n \times n}$.
- **2.y** (*Optional research project, do not submit*) In class, we played around with a visualization of our topological proof of the Fundamental Theorem of Algebra. Play around with this some more! What more insights can you glean from the picture about the polynomial or its roots? What if you change the polynomial? Are there any patterns or symmetries you notice in the images of various circles?