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MATH 374 : TOPOLOGY

Problem Set 2 – due Friday, September 23rd

INSTRUCTIONS:

If this is your week to write, please submit this assignment via Glow (to be published soon!) by 4pm on Friday; your solutions should be written in LaTeX. If this is your oral week, please be prepared by Friday to present your solutions orally (but you do not have to write them up in any form). If you have any questions (either about math or about LaTeX), please don't hesitate to reach out to me and we can figure it out.

2.1 Metrics on $\mathbb{R}^2$.

(a) Prove that the Euclidean metric on $\mathbb{R}^2$ is, in fact, a metric.

(b) Suppose $\mathcal{O}$ is a subset of $\mathbb{R}^2$ that's open with respect to the Euclidean metric. Must it also be open with respect to the taxicab metric?

(c) The Euclidean and taxicab metrics on $\mathbb{R}^2$ both have the form

$$d_p(x, y) := \left(|x_1 - y_1|^p + |x_2 - y_2|^p\right)^{1/p}$$

(d_1 is the taxicab metric, d_2 is the Euclidean metric). It turns out that d_p is a metric for every real number $p \geq 1$. (Don't worry about proving it here, although it is a fun challenge to think about when you have some spare time.) Can you describe any of the other metrics on $\mathbb{R}^2$ that we've encountered (chessboard, British Rail, and discrete) in terms of d_p?

2.2 In class we saw an example of a set that was open in $\mathbb{R}$, but not when viewed within $\mathbb{R}^2$. The following questions are inspired by Levi.

(a) If $A \subseteq \mathbb{R}$ (with respect to the Euclidean metric) is open, is it possible for $A \times \{1\}$ to be open in $\mathbb{R}^2$ (with respect to the Euclidean metric)? Either prove that it's never possible, or present an example where it is possible.

(b) If $A \subseteq \mathbb{R}$ (with respect to the Euclidean metric) is closed, is $A \times \{1\}$ closed in $\mathbb{R}^2$ (with respect to the Euclidean metric)? Either prove that this is always the case, never the case, or that it is sometimes the case and sometimes not by giving explicit examples.

2.3 Let $\mathbb{R}^\infty$ be the set of all sequences of real numbers. Determine (with proof) whether each of the following is a metric on $\mathbb{R}^\infty$.

(a) $d(x, y) := \max\{|x_n - y_n|\}$.

(b) $d(x, y) := \begin{cases} \frac{1}{n+1} & \text{if } \exists n \geq 0 \text{ s.t. } x_i = y_i \text{ for all } i \leq n \text{ and } x_{n+1} \neq y_{n+1} \\ 0 & \text{if } x = y. \end{cases}$

(c) $d(x,y) := \sum_{n=1}^{\infty} \frac{1}{2^n}\big(1-\delta(x_n,y_n)\big)$ where $\delta(x,y)=1$ if $x=y$ and 0 otherwise.

2.4 Given finite sets $A, B \subseteq \mathbb{Z}$, let $A - B := \{a - b : a \in A, b \in B\}$ and define the *Ruzsa distance* by

$$d(A,B) := \log \frac{|A-B|}{\sqrt{|A|}\sqrt{|B|}}.$$

(Here $|S|$ denotes the number of elements in a set S.) Is this a metric?

2.5 Given a metric space (X, d) where X has at least 2 elements. Prove that there exists a metric on X that's neither the discrete metric nor equal to the metric d.

2.6 Prove that a point cannot be simultaneously in the interior of A and on the boundary of A. (More generally, prove that A doesn't contain its boundary iff A consists of interior points.) Why doesn't this contradict our bizarre example from Lecture 3, in which we saw that $[0,3)$ is open in $\mathbb{R}_{\geq 0}$ with respect to the Euclidean metric?

2.7 Recall our graph theoretic example of a metric space: $\{A, B, C, D\}$ with the distance between any two of A, B, C being 2 and the distance between D and any one of A, B, C being 1.

(a) What's $\partial\{B\}$?

(b) Describe all the open sets in this space.

(c) Describe all the closed sets in this space.

2.8 Recall from class that $\mathcal{B}_r(p)$ denotes the "open ball" of radius r around the point p.

(a) Prove that for any $p \in X$ and any positive r, $\mathcal{B}_r(p)$ is open.

(b) What if $r = 0$ above? Is $\mathcal{B}_0(p)$ open? You must either prove that it's always open, prove that it's never open, or provide examples to show that it can be sometimes open and sometimes not open.

2.x (*Optional challenge problem—won't be graded*) Let $M_{n\times n}$ denote the space of all $n \times n$ matrices with real entries. Prove that $d(x,y) := \operatorname{rank}(x-y)$ is a metric on $M_{n\times n}$.

2.y (*Optional research project, do not submit*) In class, we played around with a visualization of our topological proof of the Fundamental Theorem of Algebra. Play around with this some more! What more insights can you glean from the picture about the polynomial or its roots? What if you change the polynomial? Are there any patterns or symmetries you notice in the images of various circles?