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MATH 374 : TOPOLOGY

Problem Set 3 – due Friday, September 30th

INSTRUCTIONS:

If this is your week to write, please submit this assignment via Glow by 4pm on Friday; the solutions to at least four of the problems should be written in LaTeX. If this is your oral week, please be prepared by Friday to present your solutions orally (but you do not have to write them up in any form; during our meetings you won't be using any notes). If you have any questions—either about math or about LaTeX—please don't hesitate to reach out to me and we can figure it out.

3.1 The topology of $\mathbb{R}^2$ induced by the Euclidean metric is called the *usual topology* on $\mathbb{R}^2$. Describe the topology of $\mathbb{R}^2$ induced by the taxicab metric. What about the chessboard metric? What about the British Rail metric?

3.2 Suppose $\mathcal{T}$ is a topology on $\mathbb{R}^2$ that contains the set $\{(x,x) : x \in \mathbb{R}\}$ and also contains $(x, x+2) \times \{y\}$ for each $x, y \in \mathbb{R}$. Here (x,x) denotes a point in the plane, while $(x, x+2)$ denotes an open interval!

(a) Is the interval $(\frac{3}{4}, 1) \times \{0\} \in \mathcal{T}$?

(b) Is the interval $(1,4) \times \{0\} \in \mathcal{T}$?

(c) Does $\mathcal{T}$ contain an element consisting of **countably** infinitely many points?

3.3 Let $\overline{\,\cdot\,}$ be a closure operator on X. Prove that $A \subseteq B \subseteq X$ implies $\overline{A} \subseteq \overline{B}$. [*You may not use properties of closed sets for this problem, since we used this as a lemma in class to prove properties of closed sets!*]

3.4 In class, Sasha made a very reasonable proposal that the closure of a singleton set (i.e. a set consisting of a single element) should be itself. Sadly, topology cares little for our intuition.

Construct an example of a topological space $(X, \mathcal{T})$—i.e. a space X with a topology $\mathcal{T}$ satisfying our three conditions (*O-i*)–(*O-iii*) from class—in which $2 \le |X| < \infty$, and $\overline{\{x\}} \neq \{x\}$ for some $x \in X$. Can you construct such an example in which X is infinite?

3.5 Prove that the intersection of any collection of topologies on X is a topology on X.

3.6 Recall from Lecture 5 that to any closure operator $\overline{\,\cdot\,}$ on X we associated the set $\mathcal{C}$, consisting of all $A \subseteq X$ satisfying $A = \overline{A}$. (The elements of $\mathcal{C}$ are called the *closed sets* of X induced by $\overline{\,\cdot\,}$.) We then proved that $\mathcal{C}$ must satisfy three properties:

(*C-i*) $\emptyset, X \in \mathcal{C}$,

(*C-ii*) $\mathcal{C}$ is closed under finite unions, and

(*C-iii*) $\mathcal{C}$ is closed under arbitrary intersections.

[NB: the word *closed* in (ii) and (iii) is unrelated to the term *closed set*.]

(a) In class we asserted that the process can be run in reverse, as well: given a set $\mathcal{C} \subseteq \mathcal{P}(X)$, we followed Rauan's idea and defined a closure operator on X by setting the closure of S to be the smallest element of $\mathcal{C}$ containing S. Prove that if $\mathcal{C}$ fails to satisfy any one of (C-i), (C-ii), or (C-iii), then the induced 'closure' can fail to be a closure.

(b) Verify that if $\mathcal{C}$ does satisfy all of (C-i), (C-ii), and (C-iii), then the induced operator described above *is* a closure.

(c) Prove that if you start with a closure, generate $\mathcal{C}$ as described above, and then use $\mathcal{C}$ to induce a closure, you end up with the same closure operator you started with.

(d) Prove that if you start with a set $\mathcal{C}$ satisfying properties (C-i), (C-ii), and (C-iii), induce a closure, and then use that closure to induce a set of closed sets, this set is $\mathcal{C}$.

(e) Prove that there's a bijection between the set of all possible closures on X and the set of all possible topologies on X. (Your proof should be quite short.)

3.7 Given a set X and a closure operator $\overline{\ \cdot\ }$ on X. One of the defining properties of closure is that it's *idempotent*: applying it repeatedly produces the same result as applying it once. In particular, starting with a set $A \subseteq X$ one can generate at most 2 distinct sets using the closure operator: A and $\overline{A}$. The purpose of this exercise is to explore the relationship between the closure and complement operators.

(a) Do the closure and complement operators commute? In other words, given $A \subseteq X$, does $\overline{A^c} = \overline{A}^{\,c}$?

(b) Define a new operator $i : \mathcal{P}(X) \to \mathcal{P}(X)$ by $i(A) := \left(\overline{A^c}\right)^c$. Can you give an intuitive description of this set? (Think in $\mathbb{R}^2$!)

(c) Given $A \subseteq X$, prove that there are only finitely many different sets that can be generated from A by applying complements and closures. Can you get an explicit upper bound on how many?
[*Hint: Your upper bound shouldn't depend on A or X!*]