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MATH 374 : TOPOLOGY

Problem Set 3 – due Thursday, September 26th

INSTRUCTIONS:

You should aim to submit this assignment to me in person at the start of Thursday's class; if you cannot make it to class, email me by 11am on Thursday and we can discuss alternative ways to submit your assignment. Late assignments can be left in the mailbox outside my office until 4pm on Friday (incurring a small penalty, as described in the course syllabus). Assignments will not be accepted after 4pm on Friday.

- **3.1** Consider the space $X := \{f : [0, 1] \to \mathbb{R}, \text{ a continuous function}\}$.
 - (a) Explain (with examples) why $\Delta(f,g) := \max_{t \in (0,1)} |f(t) g(t)|$ is not a metric on X.
 - (b) Prove that $d(f,g) := \max_{t \in [0,1]} |f(t) g(t)|$ is a metric on X.
 - (c) Prove that any function in X is completely determined by its behavior on \mathbb{Q} . In other words, show that if $f, g \in X$ and f(q) = g(q) for all $q \in \mathbb{Q}$, then f = g.
- **3.2** Given a function $f: X \to Y$, where X and Y are metric spaces. We proved in class that f is continuous on X if and only if $f^{-1}(\mathcal{B})$ is open in X for every open ball \mathcal{B} in Y. Use this to prove the following

Theorem. f is continuous on X if and only if $f^{-1}(\mathcal{O})$ is open in X for every open set \mathcal{O} in Y.

(In words: a function is continuous iff the preimage of any open set is open.)

- **3.3** In class we saw that any collection of disjoint open intervals must be countable. Does this also hold for closed intervals?
- **3.4** In class we described the Cantor set and some of its properties. Here we explore this topic more carefully. First, we recall the construction of the Cantor set. (This was done in class using less formal language). We begin with the open interval $\mathcal{O}_1 := (1/3, 2/3)$. Next, for each $n \ge 1$ define

$$\mathcal{O}_{n+1} := \left(rac{1}{3} \cdot \mathcal{O}_n\right) \cup \left(rac{2}{3} + rac{1}{3} \cdot \mathcal{O}_n\right),$$

where $\alpha \cdot X := \{\alpha x : x \in X\}$ and $\beta + Y := \{\beta + y : y \in Y\}$. Finally, set

$$\mathcal{C} := [0,1] \setminus \left(\bigcup_{n=1}^{\infty} \mathcal{O}_n \right).$$

It immediately follows that C is closed and bounded, hence that C is *compact*. (We'll revisit this concept in the general setting of topological spaces.)

(a) Prove that \mathcal{C} has empty interior, i.e. that no points of \mathcal{C} are interior points.

- (b) Prove that \mathcal{C} has no isolated points.
- (c) The set $\bigcup_{n=1}^{\infty} \mathcal{O}_n$ is the union of disjoint open intervals. Prove that the sum of all the lengths of all these intervals is 1. (In other words, \mathcal{C} has zero length!)
- (d) Prove that $x \in \mathcal{C}$ iff x has a ternary (i.e. base 3) expansion that doesn't use the digit 1 anywhere.
- (e) Given sets \mathcal{A} and \mathcal{B} of real numbers, define their sum and difference to be

$$\mathcal{A} + \mathcal{B} := \{a + b : a \in \mathcal{A}, b \in \mathcal{B}\} \qquad \qquad \mathcal{A} - \mathcal{B} := \{a - b : a \in \mathcal{A}, b \in \mathcal{B}\}.$$

Prove that $\mathcal{C} + \mathcal{C} = [0, 2]$ and $\mathcal{C} - \mathcal{C} = [-1, 1]$.

- **3.5** This problem builds on the previous one and introduces the notorious **Cantor-Lebesgue function**. This is a function $F : [0,1] \rightarrow [0,1]$ with the seemingly paradoxical properties that
 - F is continuous everywhere on [0, 1].
 - F(0) = 0 and F(1) = 1.
 - The measure (i.e. total length) of the set $\{x \in [0,1] : F'(x) = 0\}$ equals 1!
 - (a) Consider $f: \mathcal{C} \to [0,1]$ defined by

$$f(x) := \sum_{k=1}^{\infty} \frac{a_k/2}{2^k}$$

where $x = 0.a_1a_2a_3...$ is a ternary expansion of x that doesn't use the digit 1. Prove that f is well-defined and continuous on C, and that f(0) = 0 and f(1) = 1.

- (b) Prove that f is surjective. [Note that, bizarrely, f maps a measure 0 set onto a set of measure 1.]
- (c) Prove that if $a, b \in \mathcal{C}$ and $(a, b) \subset [0, 1] \setminus \mathcal{C}$, then f(a) = f(b).
- (d) Deduce the existence of the Cantor-Lebesgue function.