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MATH 374 : TOPOLOGY

Problem Set 4 - due Thursday, October 3rd

INSTRUCTIONS:

You should aim to submit this assignment to me in person at the start of Thursday's class; if you cannot make it to class, email me by 11am on Thursday and we can discuss alternative ways to submit your assignment. Late assignments can be left in the mailbox outside my office until 4pm on Friday (incurring a small penalty, as described in the course syllabus). Assignments will not be accepted after 4pm on Friday.

- **4.1** The topology of \mathbb{R}^2 induced by the Euclidean metric is called the *usual topology* on \mathbb{R}^2 . Describe the topology of \mathbb{R}^2 induced by the taxicab metric. What about the chessboard metric? What about the British Rail metric?
- 4.2 Note that the basis we gave for \mathbb{R}_{usual} (the collection of all bounded open intervals) has uncountably many sets in it. Find a *countable* basis of \mathbb{R}_{usual} .
- **4.3** Suppose \mathcal{T} is a topology on \mathbb{R}^2 that contains the set of points $\{(x, x) : x \in \mathbb{R}\}$, and also contains the line segments $(x, x + 2) \times \{y\}$ for each $x, y \in \mathbb{R}$. (Here (x, x) denotes a point in the plane, while (x, x + 2) denotes an open interval.)
 - (a) Is the interval $(\frac{3}{4}, 1) \times \{0\} \in \mathcal{T}$?
 - (b) Is the interval $(1, 4) \times \{0\} \in \mathcal{T}$?
 - (c) Does \mathcal{T} contain an element consisting of **countably** infinitely many points?
- **4.4** Let $\overline{\cdot}$ be a closure operator on X. Prove that $A \subseteq B \subseteq X$ implies $\overline{A} \subseteq \overline{B}$. [You may not use properties of closed sets for this problem, since we used this as a lemma in class to prove properties of closed sets!]
- **4.5** In class, Daniel made the very reasonable proposal that the closure of a singleton set (i.e. a set consisting of a single element) should be itself. Sadly, topology cares little for our intuition.

Construct an example of a topological space (X, \mathcal{T}) —i.e. a space X with a topology \mathcal{T} satisfying our three conditions from class—in which $2 \leq |X| < \infty$, and $\overline{\{x\}} \neq \{x\}$ for some $x \in X$. Can you construct such an example in which X is infinite?

- **4.6** Prove that the intersection of any collection of topologies on X is a topology on X.
- **4.7** Recall from class that given any closure operator $\overline{\cdot}$ on X, we can define what it means for a set to be closed: we say $A \subseteq X$ is *closed* iff $A = \overline{A}$. Let \mathcal{C} denote the collection of all subsets of X that are closed (with respect to a given closure operator). In class we proved that \mathcal{C} must satisfy three properties:
 - $(C-i) \ \emptyset, X \in \mathcal{C},$
 - (C-ii) C is closed under finite unions, and
 - (C-iii) C is closed under arbitrary intersections.

- [NB: the word *closed* in (ii) and (iii) is unrelated to the term *closed set*!]
- (a) In class we asserted that the process can be run in reverse, as well: given a set $C \subseteq \mathcal{P}(X)$, we can define a closure operator on X by setting the closure of S to be the smallest element of C containing S. Prove that if C fails to satisfy any one of (C-i), (C-ii), or (C-iii), then the induced 'closure' can fail to be a closure.
- (b) Verify that if C does satisfy all of (C-i), (C-ii), and (C-iii), then the induced operator described above *is* a closure.
- (c) Prove that if you start with a closure, generate C as described above, and then use C to induce a closure, you end up with the same closure operator you started with.
- (d) Prove that if you start with a set C satisfying properties (C-i), (C-ii), and (C-iii), induce a closure, and then use that closure to induce a set of closed sets, this set is C.
- (e) Deduce that there's a bijection between the set of all possible closures on X and the set of all possible topologies on X. (Your proof should be quite short.)
- **4.8** Given a set X and a closure operator $\overline{}$ on X. One of the defining properties of closure is that it's *idempotent*: applying it repeatedly produces the same result as applying it once. In particular, starting with a set $A \subseteq X$ one can generate at most 2 distinct sets using the closure operator: A and \overline{A} . The purpose of this exercise is to explore the relationship between the closure and complement operators.
 - (a) Do the closure and complement operators commute? In other words, given $A \subseteq X$, does $\overline{A^c} = \overline{A}^c$?
 - (b) Define a new operator $i : \mathcal{P}(X) \to \mathcal{P}(X)$ by $i(A) := (\overline{A^c})^c$. Can you give an intuitive description of this set? (Think in \mathbb{R}^2 !)
 - (c) Given A ⊆ X, prove that there are only finitely many different sets that can be generated from A by applying complements and closures. Can you get an explicit upper bound on how many?
 [*Hint: Your upper bound shouldn't depend on A or X!*]