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MATH 374 : TOPOLOGY

Problem Set 4 – due Friday, October 7th

INSTRUCTIONS:

If this is your week to write, please submit this assignment via Glow by 4pm on Friday; the solutions to at least four of the problems should be written in LaTeX. If this is your oral week, please be prepared by Friday to present your solutions orally (but you do not have to write them up in any form; during our meetings you won't be using any notes). If you have any questions—either about math or about LaTeX—please don't hesitate to reach out to me and we can figure it out.

4.0 Read Chapters 2 and 5 of Ivan's notes.

4.1 In Chapter 2 of Ivan's notes (*Bases of topologies*), a basis on X is defined to be any set $\mathcal{B} \subseteq \mathcal{P}(X)$ satisfying two conditions:

- $\mathcal{B}$ covers X
- For any $S, T \in \mathcal{B}$ and any $\alpha \in S \cap T$, $\exists A \in \mathcal{B}$ such that $\alpha \in A \subseteq S \cap T$.

The first condition is identical to the definition we gave in class, but the second looks different. Prove that this is equivalent to the definition we gave in class.

4.2 Prove that the topology defined by Definition 2.6 in Chapter 2 of Ivan's notes is the same as the topology generated by $\mathcal{B}$, as defined in our class.

4.3 Prove that collection of all open balls in $\mathbb{R}^2$—i.e. all sets of the form $\mathcal{B}_\delta(x)$, with respect to the Euclidean metric—is a basis on $\mathbb{R}^2$, and that it generates the usual topology $\mathbb{R}^2$.

4.4 All the parts of this question concern the Sorgenfrey line.

(a) Prove that the interval $(0, 1)$ is open.

(b) Is $(0, 1]$ open?

(c) Prove that singletons are their own closures.

(d) Prove that there does not exist a countable basis that generates the lower limit topology.

4.5 Let $\mathcal{B}$ be the set of bi-infinite arithmetic progressions consisting of integers, and let $\mathcal{T}$ denote the topology on $\mathbb{Z}$ generated by $\mathcal{B}$. (This is the *Furstenberg topology* on $\mathbb{Z}$ that we used to prove the infinitude of primes.)

(a) Prove that $\mathcal{B}$ is a basis on $\mathbb{Z}$.

(b) Let $a_n := 2^n 3^{n-1} 5^{n-2} \cdots p_{n-1}^2 p_n$, where p_k denotes the k^{th} prime number; the sequence a_n begins $2, 12, 360, 75600, \ldots$ Does this sequence converge in $\mathbb{Z}$ under the Furstenberg topology?

4.6 All parts of this question concern $\mathbb{R}_{\text{cofinite}}$.

(a) Let $a_n := 1$ for all n. What's $\lim_{n\to\infty} a_n$?

(b) Let $b_n := (-1)^n$ for all n. What's $\lim_{n\to\infty} b_n$?

(c) Let $c_n := 1 + 2 + 3 + \cdots + n$. Prove that $c_n \to -\frac{1}{12}$.

(d) Can you give a simple and complete description of convergence in $\mathbb{R}_{\text{cofinite}}$?