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Problem Set 5 – due Friday, October 14th

INSTRUCTIONS:

If this is your week to write, please submit this assignment via Glow by 4pm on Friday; the solutions to at least four of the problems should be written in LaTeX. If this is your oral week, please be prepared by Friday to present your solutions orally (but you do not have to write them up in any form; during our meetings you won't be using any notes). If you have any questions—either about math or about LaTeX—please don't hesitate to reach out to me and we can figure it out.

5.1 In class, I asserted without proof that a space is T_1 iff singletons are closed. In fact, more is true! Given a topological space $(X, \mathcal{T})$, prove that

$$(X, \mathcal{T}) \text{ is } T_1 \iff \{x\} \text{ is closed } \forall x \in X \iff \text{all finite sets are closed} \iff \forall A \subseteq X, A = \bigcap_{\substack{\mathcal{O} \text{ s.t.} \\ A \subseteq \mathcal{O} \in \mathcal{T}}} \mathcal{O}$$

5.2 In class we proved that if a topological space is Hausdorff, then every convergent sequence has a unique limit. The goal of this exercise is to show that the converse of this fails to hold.

(a) Prove that in $(\mathbb{R}, \mathcal{T}_{\text{cocountable}})$, every convergent sequence has a unique limit.
(b) Prove that $(\mathbb{R}, \mathcal{T}_{\text{cocountable}})$ isn't Hausdorff.

5.3 Last class we invented a definition of continuity in the context of topological spaces: given a function $f : X \to Y$ from a topological space $(X, \mathcal{T})$ to a topological space $(Y, \mathcal{S})$, we said that f is *continuous* at $\alpha \in X$ iff for any open set $\mathcal{O}_1 \in \mathcal{S}$ containing $f(\alpha)$, there exists an open set $\mathcal{O}_2 \in \mathcal{T}$ containing α such that $\mathcal{O}_2 \subseteq f^{-1}(\mathcal{O}_1)$. The goal of this exercise is to explore this concept further.

(a) It turns out there's an easier way to write our definition: prove that f is continuous at α iff α is an interior point of $f^{-1}(\mathcal{O})$ for every open $\mathcal{O} \ni f(\alpha)$.
(b) Suppose f is continuous on X (i.e. continuous at every point of X). Prove that whenever $p \in \mathcal{O} \in \mathcal{S}$, there must exist $\Omega_p \in \mathcal{T}$ such that $f^{-1}(p) \subseteq \Omega_p \subseteq f^{-1}(\mathcal{O})$.
(c) Deduce that if f is continuous on X, then $f^{-1}(\mathcal{O})$ must be open for any open set $\mathcal{O}$.
(d) Prove that f is continuous everywhere on X iff the preimage of every open set is open. (The *preimage* of a set A means $f^{-1}(A)$, i.e. all the points of X that get mapped into A.)

5.4 Given a topological space $(X, \mathcal{T})$, any subset $A \subseteq X$ inherits a natural topology, called the *subspace topology* on A:

$$\mathcal{T}_{\text{subspace}} := \{\mathcal{O} \cap A : \mathcal{O} \in \mathcal{T}\}.$$

Show that the subspace topology A inherits from X is the coarsest topology on A such that i is continuous on A, where $i : A \to X$ is defined $i(x) := x$.

5.5 Given two continuous functions $f, g : X \to Y$ where X is a topological space and Y is a Hausdorff space.

(a) Suppose $A \subseteq X$ and $f(a) = g(a)$ for all $a \in A$. Prove that $f(x) = g(x)$ for all $x \in \overline{A}$.
(b) Prove that the set $\{x \in X : f(x) = g(x)\}$ is closed.