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MATH 374 : TOPOLOGY

Problem Set 5 - due Friday, October 11th

INSTRUCTIONS:

You should aim to submit this assignment to me in person at the start of Thursday's class. As usual, assignments can be left in the mailbox outside my office until 4pm on Friday; unlike usual, there will be no penalty for submitting after lecture this week. Assignments will not be accepted after 4pm on Friday, unless it's Mountain Day, in which case the hard deadline will shift by 24 hours to 4pm on Saturday.

- 5.0 Read Chapters 2 and 5 of Ivan's notes.
- **5.1** In Chapter 2 of Ivan's notes (*Bases of topologies*), a basis on X is defined to be any set $\mathcal{B} \subseteq \mathcal{P}(X)$ satisfying two conditions:
 - \mathcal{B} covers X
 - For any $S, T \in \mathcal{B}$ and any $\alpha \in S \cap T$, $\exists A \in \mathcal{B}$ such that $\alpha \in A \subseteq S \cap T$.

The first condition is identical to the definition we gave in class, but the second looks different. Prove that this is equivalent to the definition we gave in class.

- 5.2 Prove that the topology defined by Definition 2.6 in Chapter 2 of Ivan's notes is the same as the topology generated by \mathcal{B} , as defined in our class.
- **5.3** Prove that collection of all open balls in \mathbb{R}^2 —i.e. all sets of the form $\mathcal{B}_{\delta}(x)$, with respect to the Euclidean metric—is a basis on \mathbb{R}^2 , and that it generates the usual topology \mathbb{R}^2 .
- 5.4 All the parts of this question concern the Sorgenfrey line.
 - (a) Prove that the interval (0, 1) is open.
 - (b) Is (0, 1] open?
 - (c) Prove that singletons are their own closures.
 - (d) Prove that there does not exist a countable basis that generates the lower limit topology. [*Hint: In view of problem 4.2, make sure your proof doesn't apply to* \mathbb{R}_{usual} .]
- **5.5** Let \mathcal{B} be the set of bi-infinite arithmetic progressions consisting of integers, and let \mathcal{T} denote the topology on \mathbb{Z} generated by \mathcal{B} . (This is the *Furstenberg topology* on \mathbb{Z} that we used to prove the infinitude of primes.)
 - (a) Prove that \mathcal{B} is a basis on \mathbb{Z} .
 - (b) Let $a_n := 2^n 3^{n-1} 5^{n-2} \cdots p_{n-1}^2 p_n$, where p_k denotes the k^{th} prime number; the sequence a_n begins 2, 12, 360, 75600, ... Does this sequence converge in \mathbb{Z} under the Furstenberg topology? If not, prove it; if so, find all values it converges to, and prove that your list is exhaustive.

5.6 In class, I asserted without proof that a space is T_1 iff singletons are closed. In fact, more is true! Given a topological space (X, \mathcal{T}) , prove that

 $(X,\mathcal{T}) \text{ is } T_1 \iff \{x\} \text{ is closed } \forall x \in X \iff \text{ all finite sets are closed } \iff \forall A \subseteq X, A = \bigcap_{\substack{\mathcal{O} \text{ s.t.} \\ A \subseteq \mathcal{O} \in \mathcal{T}}} \mathcal{O}$

- **5.7** All parts of this question concern $\mathbb{R}_{\text{cofinite}}$.
 - (a) Let $a_n := 1$ for all n. What's $\lim_{n \to \infty} a_n$?
 - (b) Let $b_n := (-1)^n$ for all n. What's $\lim_{n \to \infty} b_n$?
 - (c) Let $c_n := 1 + 2 + 3 + \dots + n$. Prove that $c_n \to -\frac{1}{12}$.
 - (d) Can you give a simple and complete description of convergence in $\mathbb{R}_{cofinite}$?