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Problem Set 7 – due Friday, November 4th

INSTRUCTIONS:

If this is your week to write, please submit this assignment via Glow by 4pm on Friday; the solutions to at least three of the problems should be written in LaTeX. If this is your oral week, please be prepared by Friday to present your solutions orally (but you do not have to write them up in any form; during our meetings you won't be using any notes). If you have any questions—either about math or about LaTeX—please don't hesitate to reach out to me and we can figure it out.

7.0 Check out the proof of (a special case of) the Heine-Borel on the course website.

7.1 Suppose X is a compact topological space, and consider the subspace $A \subseteq X$.

(a) Show by example that A need not be compact.
(b) Prove that if A is closed, then it must be compact.

7.2 Use the fact that $[0,1]$ is compact to prove that $[0,1] \times [0,1]$ is compact.

7.3 For each space, decide (with proof) whether or not it's compact.

(a) D^2 (the closed disk in $\mathbb{R}^2$)
(b) S^2 (the (hollow) sphere in $\mathbb{R}^3$)
(c) $\mathbb{Z}$ under the Furstenberg topology.
(d) $\mathbb{R}_7$ (i.e. under the particular point topology).
(e) $\mathbb{R}/\sim$, where $x \sim y$ iff $x - y \in \mathbb{Z}$.
(f) $\mathbb{R}_{\text{ray}}$
(g) $\mathbb{Q}_{\text{usual}}$
(h) The interval $[0,1)$ as a subspace of $\mathbb{R}_{\text{sorgenfrey}}$

7.4 Find an example of a topological space X and two compact subspaces $A, B \subseteq X$ such that $A \cap B$ is *not* compact.

7.5 Suppose $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ is a homeomorphism, and say $\mathcal{B}$ is a basis for $\mathcal{T}$. Prove that $\{f(A) : A \in \mathcal{B}\}$ is a basis of $\mathcal{S}$.

7.6 For each of the pairs of spaces below, decide whether or not they are homeomorphic.

(a) S^1 and D^2
(b) $\mathbb{R}$ and $\mathbb{R}^2$ (both under the usual topology)
(c) $\mathbb{Q}$ and $\mathbb{Q}^2$ (both under the usual topology)