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Problem Set 8 – due Friday, November 11th

INSTRUCTIONS:

If this is your week to write, please submit this assignment via Glow by 4pm on Friday; the solutions to **8.1** and **8.2** should be written in LaTeX. If this is your oral week, please be prepared by Friday to present your solutions orally (but you do not have to write them up in any form; during our meetings you won't be using any notes). If you have any questions—either about math or about LaTeX—please don't hesitate to reach out to me and we can figure it out.

8.1 The goal of this exercise is to establish that $(\mathbb{Z}, \mathcal{T}_{\text{Furstenberg}})$ is *metrizable*, i.e. that there exists a metric on $\mathbb{Z}$ that induces the Furstenberg topology. (In fact, there exist many such metrics!) Set

$$d(a,b) = 1 - \sum_{n \mid a-b} \frac{1}{2^n}$$

where the sum runs over all positive integers n dividing $a - b$. For example, $d(2,6) = \frac{3}{16}$.

(a) Prove that d is a metric on $\mathbb{Z}$.

(b) Prove that any nonempty open ball with respect to d contains a bi-infinite arithmetic progression.

(c) Prove that any bi-infinite arithmetic progression contains a nonempty open ball with respect to d.

(d) Prove that the metric d induces the Furstenberg topology on $\mathbb{Z}$.

8.2 We explore properties of the Sorgenfrey line.

(a) Prove that $\mathbb{R}_{\text{sorgenfrey}}$ is totally disconnected (i.e. its only connected subsets are singletons and $\emptyset$).

(b) Prove that $\mathbb{R}_{\text{sorgenfrey}}$ is T_1.

(c) Prove that $\mathbb{R}_{\text{sorgenfrey}}$ is normal. [*Hint: Given two closed sets, find an open cover of each.*]

(d) (**Challenge problem, not for submission**) The Sorgenfrey line was historically important because it provided the first example that normality isn't preserved under products. Consider the line $X := \{(x, -x) : x \in \mathbb{R}\}$ as a subspace of the Sorgenfrey plane $\mathbb{R}^2_{\text{sorgenfrey}}$. We can partition this into two sets:

$$A := X \cap \mathbb{Q}^2 \qquad B := X \cap (\mathbb{R} \setminus \mathbb{Q})^2.$$

Show that A and B are closed, but that any two open sets containing A and B must intersect.

8.3 Recall that a space is *Hausdorff* iff any two distinct points live in disjoint opens; a space is *normal* iff any two disjoint closed sets can be enlarged to two disjoint open sets. There's a natural property in between Hausdorffness and normality:

Definition. A space is *regular* iff for any point p and any closed set C not containing p, there exist disjoint open sets $\mathcal{O}$ and $\mathcal{O}'$ such that $p \in \mathcal{O}$ and $C \subseteq \mathcal{O}'$.

The purpose of this exercise is to explore the relationships between our various separation axioms.

(a) Show that if a space is both regular and T_0 then it must be Hausdorff.

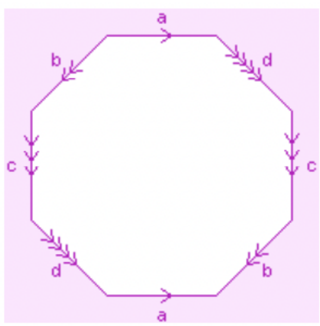
(b) Give an example of a regular space that's not Hausdorff.

(c) Prove that if X is compact and Hausdorff then it must be regular. [*Hint: Given a point p and a closed set C, try to separate p from every point of C.*]

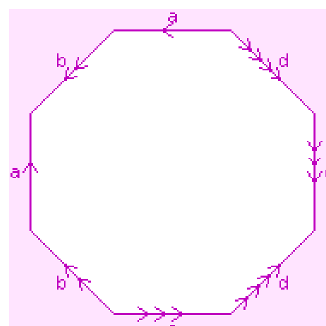
(d) Prove that if X is compact and Hausdorff then it must be normal.

(e) If a space is both normal and T_1, must it be compact and Hausdorff?

8.4 Give the simplest description you can of the surfaces represented by each of the following gluing diagrams. (You don't have to rigorously prove anything.)



Gluing diagram (a)



Gluing diagram (b)

8.5 For each of the following, give an answer along with a justification (as rigorous as possible).

(a) Is $\mathbb{R} \times \mathbb{R}$ homeomorphic to $\mathbb{R} \times \mathbb{R}_{\geq 0}$?

(b) Is $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ homeomorphic to $\mathbb{R} \times \mathbb{R}_{\geq 0}$?