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MATH 374 : TOPOLOGY

Solution Set 1

1.1 In the video linked above, a proof is presented that Hex always has a winner. Write down a clear exposition of this proof. I encourage you to change the order / terminology / notation of the proof: the goal is to write a proof that is crystal clear and can be followed without access to the video. [*As we'll see later in the course, this theorem has topological implications.*]

[Emily's solution] I will assume a fully filled-in hex board. This is justified as coloring in an additional tile after the first player has won may make the second player win (the theorem says this won't happen) but will never make it so the first player has lost.

At any intersection of tiles, there are line segments; crossing from one tile into another across the line segment may keep the color constant or switch it. Highlight all the line segments of the board which, when crossed over, switch the color. Now consider a corner of the game board; there must be a highlighted segment coming from it. Follow the highlighted path. It cannot branch, as that would require an intersection of three hexagons to have three highlighted segments, which would require three colors. It cannot terminate in the middle of the board, as that would require an intersection of three hexagons to have only one highlighted segment, implying hex A has the same color as hex B, which has the same color as hex C, which does not have the same color as hex A; this would be a contradiction. The only other possibility is that it terminates at a corner.

Since there are four corners, there must be two such paths; these cannot go to diagonally opposite corners, as the paths divide the colors, and that would force adjacent sides of the board to be the same color, which is false. Therefore, either top-left connects to top-right and bottom-left connects to bottom-right, whereupon the horizontal player wins and the vertical player loses, or top-left connects to bottom-left and top-right connects to bottom-right, whereupon the vertical player wins and the horizontal player loses.

1.2 This problem is an introduction to *projective geometry*, a beautiful area of mathematics that emerged from developments in art during the Renaissance. Fix a point O (this stands for origin, but we're going to be working without reference to coordinates). We say two points T and T' are *in perspective* iff they lie on the same line through O. (If you imagine a light source at O, two points are in perspective iff one of them lies in the other's shadow.)

Suppose ABC and $A'B'C'$ are two generic triangles in the plane that are in perspective, i.e. corresponding vertices are in perspective. (By *generic* I mean the triangles are disjoint and that none of the edges are parallel.) Let

$$X := \overleftrightarrow{AB} \cap \overleftrightarrow{A'B'} \quad , \quad Y := \overleftrightarrow{AC} \cap \overleftrightarrow{A'C'} \quad , \quad Z := \overleftrightarrow{BC} \cap \overleftrightarrow{B'C'}$$

where $\overleftrightarrow{ST}$ denotes the unique line passing through the points S and T. Click here for an interactive illustration.

Give a short explanation (no need for a formal proof) of why X, Y, Z must be collinear. [*Hint: Stare at an illustration and try to imagine the two triangles are generically positioned in* $\mathbb{R}^3$ *rather than both lying in the same plane.*]

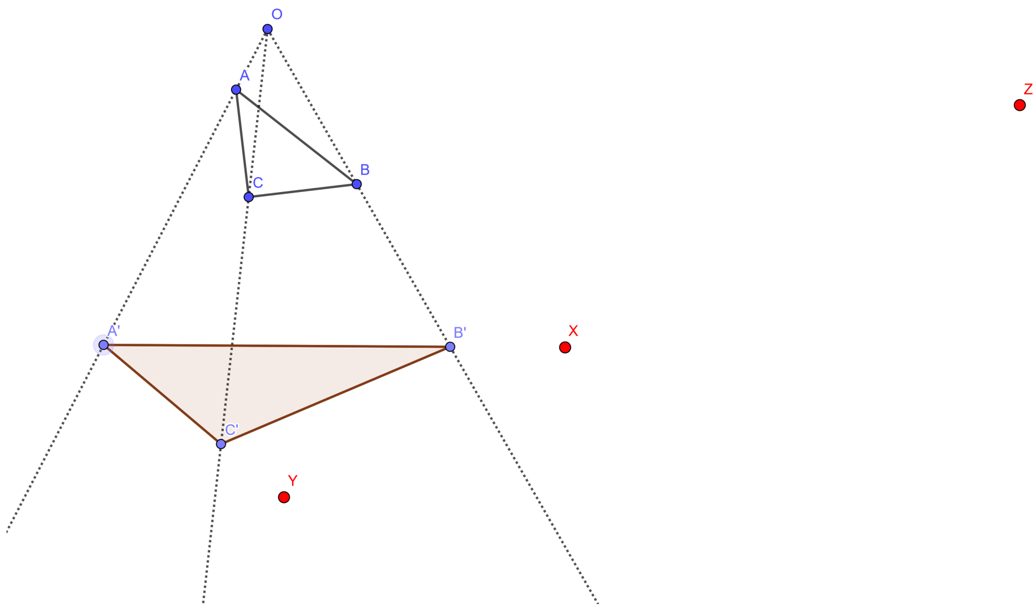


Figure described in solution

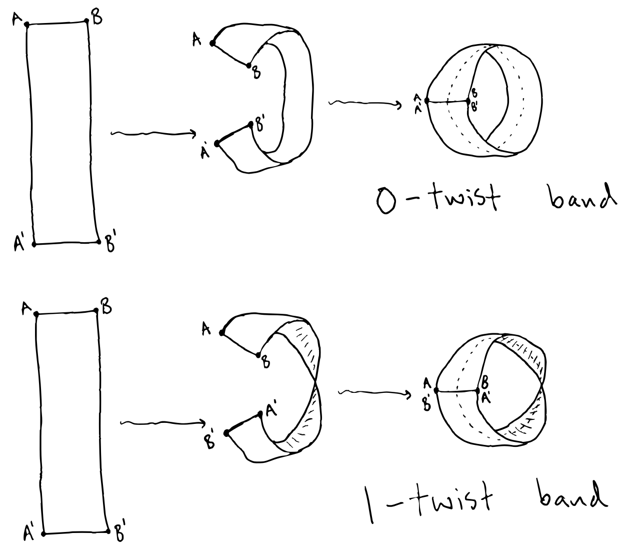
[Lizzie's solution, lightly edited] We can describe why X, Y, and Z will always be collinear by first imagining this problem in three dimensional space with each triangle on its own plane. In this case, X lies on the plane of triangle ABC (since X lies on the line $\overleftrightarrow{AB}$), but it also lies on the plane of triangle $A'B'C'$. Similarly, we see that Y and Z also lie on the intersection of these two planes, and since the intersection of two planes is a line, X, Y, Z must be collinear.

To bring the problem back into the original two-dimensional case, we will start with the three-dimensional case and perform a continuous deformation into the two-dimensional case. Think of the intersection of the planes that X, Y, Z lie on as a "hinge". If we push the planes in towards each other, both rotating on our hinge, eventually we will be able to collapse the two planes into one plane containing the hinge. Since X, Y, Z lie on the hinge line, which is now in the same plane as the triangles, we see that X, Y, Z will always be collinear in two dimensions.

DISCUSSION. It turns out there's a converse to this theorem as well; together, these form Desargues' theorem, a prototype in the development of projective geometry.

1.3 Using a rectangular strip of paper, one can form a bracelet by bending the strip and gluing the edges (without twisting the strip along the way); I'll call this a *0-twist band*. If instead we introduce one twist before gluing the edges, we obtain a *1-twist band*, popularly known as the Möbius strip. (See below for an attempt at an illustration.) Similarly one can form an *n-twist band* by introducing n twists prior to gluing the edges together.

If you cut along the line halfway across the 0-twist band (indicated by the dotted line in the illustration below), the band falls apart into two identical 0-twist bands. If instead you cut along the line 1/3 of the way across the 0-twist band, the band falls apart into two 0-twist bands, one of which is twice as thick as the other.



(a) What would happen if you cut along the line halfway across the 1-twist band? Describe in words what you think should happen. You may use drawings to help you, but **do not physically construct this yet**. (Also, as always with problem sets, please don't do any online searches.)

> A reasonable guess: get a 2-twist band that's twice as long as the original. But this isn't quite right...

(b) Having completed part (a), construct a physical model of a 1-twist band and cut it along the halfway line. Was your prediction in part (a) accurate? If not, can you now explain (in words) why you get what you get? *Please leave part (a) as you wrote it, whether or not your prediction was correct*; keep in mind that your problem set will be graded based only on effort, not on correctness.

> We get a 4-twist band! Walking along the new surface, we pass the twist in the Möbius strip twice. Where do the two additional twists come from? Consider wrapping an untwisted strip of paper twice around a cylinder and then gluing the ends together; when removed from the cylinder and shaken, you'll see that it is a 2-twist band!

(c) What would happen if you cut along the line 1/3 of the way across the 1-twist band? Describe in words what you think should happen. You may use drawings to help you, but **do not physically construct this yet**.

> You might expect to get two 1-twist bands with one of them twice as long as the other.

(d) Having completed part (c), construct a physical model of a 1-twist band and cut it along the 1/3 line. Was your prediction in part (c) accurate? If not, can you now explain (in words) why you get what you get? *Please leave part (c) as you wrote it, whether or not your prediction was correct.*

> You actually get a 1-twist band as long as the original band which is interlinked with a 4-twist band twice as long as the original; each of these bands has width $\frac{1}{3}$ of the original band. In a couple weeks we'll see how to draw such situations that don't take artistic skill, and also make transparent what to expect when you cut!