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MATH 374 : TOPOLOGY

Solution Set 2

2.1 For each of the following metrics on \mathbb{R}^2 , draw a picture the open ball $\mathcal{B}_3((2,0))$. No proofs necessary.

(a) The chessboard metric $d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.



This open ball is the interior of a square of side length 6, centered at (2,0), not including any of the boundary.

(b) The British Rail metric

$$d(x,y) := \begin{cases} |x| + |y| & \text{if } x \neq y \\ 0 & \text{otherwise.} \end{cases}$$

(Here |x| denotes the Euclidean distance from x to the origin.)



This open ball is the single point (2,0) union with the interior of the unit circle centered at the origin (not including any of the boundary).

(c) The discrete metric $d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$

By definition, $d(x, y) \leq 1 \ \forall x, y \in \mathbb{R}^2$. Thus, the open ball is all of \mathbb{R}^2 .

- **2.2** Suppose (X, d) is a metric space and $\mathcal{A} \subseteq X$. We say $p \in X$ is an *interior point* of \mathcal{A} iff $\exists r > 0$ such that $\mathcal{B}_r(p) \subseteq \mathcal{A}$, and that $p \in X$ is a *limit point* of \mathcal{A} iff there exists a sequence (a_n) of points in $\mathcal{A} \setminus \{p\}$ such that $\lim_{n \to \infty} a_n = p$. (As always, $\mathcal{B}_r(p)$ denotes the ball of radius r around p.)
 - (a) Prove that \mathcal{A} is open iff every point of \mathcal{A} is an interior point of \mathcal{A} . (In class we defined: \mathcal{A} is open iff $\partial \mathcal{A} \cap \mathcal{A} = \emptyset$.)

 (\Rightarrow) Suppose \mathcal{A} is open. Pick any $a \in \mathcal{A}$. By hypothesis, $a \notin \partial \mathcal{A}$, so there is an $\epsilon > 0$ such that $\mathcal{B}_{\epsilon}(a) \subseteq \mathcal{A}$; in other words, a is an interior point of \mathcal{A} . Thus every point of \mathcal{A} is an interior point.

 (\Leftarrow) By definition, any interior point of \mathcal{A} has a ball of some radius $\epsilon > 0$ around it such that the ball is entirely contained in \mathcal{A} , which means an interior point cannot lie on $\partial \mathcal{A}$. Thus, if every point of \mathcal{A} is an interior point, then no point of \mathcal{A} is on the boundary of \mathcal{A} . In other words, $\mathcal{A} \cap \partial \mathcal{A} = \emptyset$, whence \mathcal{A} is open.

(b) Prove that \mathcal{A} is closed iff every limit point of \mathcal{A} is in \mathcal{A} . (In class we defined: \mathcal{A} is closed iff $\partial \mathcal{A} \subseteq \mathcal{A}$.)

 (\Rightarrow) Suppose \mathcal{A} has a limit point p that is not in the set. Then for any $\epsilon > 0$, the ball of radius ϵ about p must contain a point in \mathcal{A} , by definition of p being a limit point. But that ball also contains p itself, which is in \mathcal{A}^c . Thus the ball intersects both \mathcal{A} and \mathcal{A}^c , hence p is on the boundary of \mathcal{A} . Since \mathcal{A} does not contain one of its boundary points, it is not closed. Thus any closed set must contain all its limit points.

(\Leftarrow) Suppose \mathcal{A} is not closed. Then there is some point $b \in \partial \mathcal{A} \setminus \mathcal{A}$. Since b is on the boundary of \mathcal{A} , for any $\epsilon > 0$, the ball of radius ϵ about b intersects \mathcal{A} . Let n_{ϵ} be a point in the intersection of the ball of radius ϵ and \mathcal{A} ; since $b \notin \mathcal{A}$, we see that $n_{\epsilon} \in \mathcal{A} \setminus \{b\}$. Then the sequence $n_1, n_{1/2}, n_{1/3}, \ldots$ is a sequence in \mathcal{A} converging to b, so b is a limit point of \mathcal{A} . Thus \mathcal{A} does not contain all its limit points. We conclude that any set that does contain all its limit points must be closed.

2.3 Suppose (X, d) is a metric space. Prove that $\mathcal{B}_r(p)$ is open for any $p \in X$ and any r > 0.

Pick any $q \in \mathcal{B}_r(p)$, and set $\epsilon := r - d(p,q)$. I claim

 $\mathcal{B}_{\epsilon}(q) \subseteq \mathcal{B}_{r}(p).$

To see this, pick any $x \in \mathcal{B}_{\epsilon}(q)$. Then $d(x,p) \leq d(x,q) + d(p,q) < \epsilon + d(p,q) = r$. Thus every point of $\mathcal{B}_{r}(p)$ is interior; it follows that $\mathcal{B}_{r}(p)$ is open.

- 2.4 Decide (with proof or counterexample) whether each of the following is a metric space.
 - (a) $\mathbb{R}^{\infty} := \{(a_n) : (a_n) \text{ is a sequence of real numbers}\}$, with respect to $d(x, y) := \max\{|x_n y_n|\}$. This isn't well-defined: the sequences $x_n = 0$ and $y_n = n$ would get infinitely far apart under this metric, but the codomain of any metric must be \mathbb{R} and $\infty \notin \mathbb{R}$!

DISCUSSION. The fact that some sequences diverge doesn't, by itself, guarantee that the metric is ill-defined. For example, for the divergent sequence (y_n) in the solution above, we have $d(y_n, y_n) = 0$.

(b) $\mathcal{F} := \{A \subseteq \mathbb{Z} : A \text{ is finite and nonempty}\}, \text{ with respect to } d(X,Y) := \log \frac{|X-Y|}{\sqrt{|X|}\sqrt{|Y|}}. \text{ Here } |S|$

denotes the size of S and $X - Y := \{x - y : x \in X, y \in Y\}.$

No, this is not a metric, because $\overline{d(A, A)}$ might be nonzero. For example, let $A := \{2, 3\}$. Then $A - A = \{0, \pm 1\}$, so $d(A, A) = \log \frac{3}{2} \neq 0$.

DISCUSSION. Remarkably, this function (called the *Ruzsa distance*) satisfies all the other properties, including the triangle inequality.

- **2.5** Exploring metrics on \mathbb{R}^2 .
 - (a) Prove that the Euclidean metric on \mathbb{R}^2 is, in fact, a metric.

By definition,

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

This equals 0 iff both $(x_1 - y_1)^2$, $(x_2 - y_2)^2 = 0$, which happens iff $x_1 = y_1$ and $x_2 = y_2$. It's also clear that this metric is symmetric: d(x, y) = d(y, x). It remains to prove the triangle inequality.

The most direct approach is quite algebraically involved. However, we can simplify this significantly by observing that the Euclidean distance is translation invariant, i.e. that d(x,y) = d(x-z, y-z) for any z. After translating appropriately, we see that triangle inequality is equivalent to showing that

$$d(x,y) \le d(x,0) + d(0,y)$$
(*)

for all x, y.

Note that $(x_1y_2 - x_2y_1)^2 \ge 0$. It follows that

$$(x_1y_1 + x_2y_2)^2 \le (x_1^2 + x_2^2)(y_1^2 + y_2^2). \tag{(\dagger)}$$

In particular,

$$-x_1y_1 - x_2y_2 \le \sqrt{(x_1^2 + x_2^2)(y_1^2 + y_2^2)}.$$

From here, it's straightforward to deduce

$$\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \le \sqrt{x_1^2 + x_2^2} + \sqrt{y_1^2 + y_2^2}$$

which is precisely (*).

DISCUSSION. One nice interpretation of (\dagger) is in the language of linear algebra:

 $\vec{x}\cdot\vec{y} \leq |\vec{x}|\cdot|\vec{y}|$

where the left hand side is the dot product, while the right hand side is ordinary multiplication on \mathbb{R} .

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DISCUSSION. Above, we saw that (\dagger) implies (*); it turns out that the converse implication holds as well, so the inequality (\dagger) is equivalent to the triangle inequality for the Euclidean metric on \mathbb{R}^2 . Similarly, it turns out the triangle inequality for the Euclidean metric on \mathbb{R}^n is equivalent to the following:

Lemma 1 (Cauchy-Schwarz inequality). For any real numbers a_i, b_i , we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right) \left(\sum_{i=1}^{n} b_i^2\right)$$

(or equivalently, $\vec{a} \cdot \vec{b} \leq |\vec{a}| \cdot |\vec{b}|$).

For more about this, see the document I posted on the course website, under the HW tab.

(b) Suppose \mathcal{O} is a subset of \mathbb{R}^2 that's open with respect to the Euclidean metric. Must it also be open with respect to the taxicab metric?

Suppose $\mathcal{O} \subseteq \mathbb{R}^2$ is open with respect to the Euclidean metric. Pick any $\alpha \in \mathcal{O}$; we claim that α is in the interior of \mathcal{O} with respect to the taxicab metric. Since \mathcal{O} is open with respect to the Euclidean metric, $\exists \delta > 0$ such that the open euclidean ball of radius δ around α is entirely contained inside \mathcal{O} .

Consider the open taxicab ball of radius δ around α . Pick any x in this ball; by definition, the taxicab distance between x and α is smaller than δ , i.e.

$$|x_1 - \alpha_1| + |x_2 - \alpha_2| < \delta.$$

Squaring both sides, we deduce

$$|x_1 - \alpha_1|^2 + |x_2 - \alpha_2|^2 \le |x_1 - \alpha_1|^2 + |x_2 - \alpha_2|^2 + 2|x_1 - \alpha_1| \cdot |x_2 - \alpha_2| < \delta^2.$$

This implies that x lies in the Euclidean ball of radius δ around α , which we know is entirely contained in \mathcal{O} . We've therefore shown that every point in the taxicab ball of radius δ around α is contained entirely in \mathcal{O} ; it follows that α is an interior point of \mathcal{O} with respect to the taxicab metric, as desired.

DISCUSSION. All this becomes much more clear when looking at pictures: the taxicab open ball is the largest diamond that fits inside the Euclidean ball of the same radius.

(c) The Euclidean and taxicab metrics on \mathbb{R}^2 both have the form

$$d_p(x,y) := \left(|x_1 - y_1|^p + |x_2 - y_2|^p\right)^{1/p}$$

 $(d_1 \text{ is the taxicab metric, } d_2 \text{ is the Euclidean metric})$. It turns out that d_p is a metric for every real number $p \ge 1$. (Don't worry about proving it here, although it is a fun challenge to think about when you have some spare time.) Can you describe any of the other metrics on \mathbb{R}^2 that we've encountered (chessboard, British Rail, and discrete) in terms of d_p ? No formal proofs necessary, but give a bit of justification for your answer.

This is open ended, of course, but the cleanest answers are those that describe the metric in terms of d_p without reference to specific inputs.

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Claim. The chessboard metric is $d_{\infty} := \lim_{p \to \infty} d_p$.

Proof. If x = y then $d_p(x, y) = 0$ for all p, so the limit is also 0. If $x \neq y$, then without loss of generality we have

$$\max\{|x_1 - y_1|, |x_2 - y_2|\} = |x_1 - y_1| > 0.$$

Then

$$d_p(x,y) = |x_1 - y_1| \left(1 + \left(\frac{|x_2 - y_2|}{|x_1 - y_1|} \right)^p \right)^{1/p}$$

Since this is bounded above by $|x_1 - y_1| \cdot 2^{1/p}$ and bounded below by $|x_1 - y_1|$, we see $d_p(x, y) \to |x_1 - y_1|$ as $p \to \infty$.

We can also express the **discrete metric** in terms of d_p (albeit in a more artificial form) as $\min\{\lceil d_2 \rceil, 1\}$.

DISCUSSION. The metric d_p is called the ℓ^p metric; you will explore it in virtually any advanced course on analysis.

2.6 Given a metric space (X, d) where X has at least 3 elements. Prove that there exists a metric on X that's not a scalar multiple of d or of the discrete metric.

Observe that rescaling a metric doesn't affect its metric properties. Also, it's easy to see that summing any two metrics produces a metric. Thus, any linear combination $D(x,y) := \alpha d_1(x,y) + \beta d_2(x,y)$ of any two metrics d_1, d_2 is a metric as well. In particular, if the given metric d isn't the discrete metric, then the sum of d and the discrete metric produces a new metric on X.

HOWEVER: if d is the discrete metric, then we haven't solved the problem, since in this case the sum of d and the discrete metric would be a scalar multiple of d! So in this case, we have to do something more clever. There are many approaches to this; here's one.

Given a metric d, set $D(x, y) := \frac{d(x, y)}{1+d(x, y)}$. I claim that D is a metric. It's easy to verify the first two properties, so it suffices to handle the triangle inequality:

$$\begin{aligned} D(x,z) &= 1 - \frac{1}{1 + d(x,z)} \le 1 - \frac{1}{1 + d(x,y) + d(y,z)} = \frac{d(x,y) + d(y,z)}{1 + d(x,y) + d(y,z)} \\ &= \frac{d(x,y)}{1 + d(x,y) + d(y,z)} + \frac{d(y,z)}{1 + d(x,y) + d(y,z)} \\ &\le \frac{d(x,y)}{1 + d(x,y)} + \frac{d(y,z)}{1 + d(y,z)} = D(x,y) + D(y,z). \end{aligned}$$

It's a simple exercise to check that $D \neq d$ and also cannot equal the discrete metric.

2.7 Given (X, d) a metric space and $\mathcal{A} \subseteq X$. Prove that \mathcal{A} is closed iff \mathcal{A}^c is open.

We warm up with the following useful observation:

Lemma 2. For any set $\mathcal{A} \subseteq X$, we have $\partial \mathcal{A} = \partial \mathcal{A}^c$.

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Proof of Lemma. Let $x \in \partial \mathcal{A}$. Then by the definition of a boundary point of \mathcal{A} , for any $\epsilon > 0$, we have $B_{\epsilon}(x) \cap \mathcal{A} \neq \emptyset$ and $B_{\epsilon}(x) \cap \mathcal{A}^{c} \neq \emptyset$. Since $(\mathcal{A}^{c})^{c} = \mathcal{A}$, we can just as well write that $B_{\epsilon}(x) \cap \mathcal{A}^{c} \neq \emptyset$ and $B_{\epsilon}(x) \cap (\mathcal{A}^{c})^{c} \neq \emptyset$. But this is exactly the definition for x to be in the boundary of \mathcal{A}^{c} . Thus $\partial \mathcal{A} \subseteq \partial \mathcal{A}^{c}$. Replacing \mathcal{A} with \mathcal{A}^{c} , we have that $\partial \mathcal{A}^{c} \subseteq \partial (\mathcal{A}^{c})^{c}$. Again, $(\mathcal{A}^{c})^{c} = \mathcal{A}$, so that means $\partial \mathcal{A}^{c} \subseteq \partial \mathcal{A}$. We now have subsets in both directions, so we conclude $\partial \mathcal{A} = \partial \mathcal{A}^{c}$.

Now we turn to the given problem. Suppose \mathcal{A} is open. Then $\partial \mathcal{A} \cap \mathcal{A} = \emptyset$ by definition of being an open set. That means all of $\partial \mathcal{A}$ is in \mathcal{A}^c . Since $\partial \mathcal{A} = \partial \mathcal{A}^c$ by our lemma, we have that \mathcal{A}^c contains all of its own boundary, hence it is closed. Now suppose \mathcal{A} is closed. Then it contains all of its boundary, so \mathcal{A}^c contains none of the shared boundary, hence $\mathcal{A}^c \cap \partial \mathcal{A}^c = \emptyset$ and \mathcal{A}^c is open.

DISCUSSION. Most sets that you encounter in the wild are *neither* open nor closed!

- **2.8** In class we saw an example of a set that was open in \mathbb{R} , but neither open nor closed when viewed as sitting in \mathbb{R}^2 .
 - (a) If $A \subseteq \mathbb{R}$ (with respect to the Euclidean metric) is open, is it possible for $A \times \{1\}$ to be open in \mathbb{R}^2 (with respect to the Euclidean metric)? Either prove that it's never possible, or present an example where it is possible.

No, it is not possible. To see this, pick any point in $A \times \{1\}$, say, (a, 1), and consider a ball $\mathcal{B}_r(a, 1)$ in \mathbb{R}^2 . This ball clearly intersects $A \times \{1\}$, since $(a, 1) \in \mathcal{B}_r(a, 1)$. But $\mathcal{B}_r(a, 1)$ also contains the point $(a, 1 + \frac{r}{2}) \notin A \times \{1\}$. In other words, every point in $A \times \{1\}$ is on the boundary of $A \times \{1\}$, so the set cannot be closed!

(b) If $A \subseteq \mathbb{R}$ (with respect to the Euclidean metric) is closed, is $A \times \{1\}$ closed in \mathbb{R}^2 (with respect to the Euclidean metric)? Either prove that this is always the case, never the case, or that it is sometimes the case and sometimes not by giving explicit examples.

It must be closed in \mathbb{R}^2 as well. We'll prove this by showing that $(A \times \{1\})^c$ is open.

Pick $p \notin A \times \{1\}$. If $p_2 = 1$, then $p_1 \notin A$. Since A is closed, A^c is open, so there must exist $\delta > 0$ such that $(p_1 - \delta, p_1 + \delta) \cap A = \emptyset$. It immediately follows that $\mathcal{B}_{\delta}(p) \cap (A \times \{1\}) = \emptyset$. We've proved:

$$p_2 = 1 \implies p \in \operatorname{int}((A \times \{1\})^c).$$
 (\$)

If $p \notin A \times \{1\}$ and $p_2 \neq 1$, we proceed differently. Set $\epsilon := \frac{|p_2-1|}{2}$, and consider $\mathcal{B}_{\epsilon}(p)$. For any $x \in \mathcal{B}_{\epsilon}(p)$, we have

$$|x_2 - p_2| \le \sqrt{|x_1 - p_1|^2 + |x_2 - p_2|^2} < \epsilon.$$

We deduce

$$2\epsilon = |p_2 - 1| \le |x_2 - 1| + |x_2 - p_2| < |x_2 - 1| + \epsilon$$

whence $|x_2 - 1| > \epsilon$. In particular, $x \notin A \times \{1\}$. We conclude that $\mathcal{B}_{\epsilon}(p) \cap (A \times \{1\}) = \emptyset$, or in other words, that $\mathcal{B}_{\epsilon}(p) \subseteq (A \times \{1\})^c$. We've proved:

$$p_2 \neq 1 \implies p \in \operatorname{int}((A \times \{1\})^c).$$
 (\heartsuit)

Combining (\clubsuit) and (\heartsuit) shows that $(A \times \{1\})^c$ consists entirely of interior points, hence must be open; it follows that $A \times \{1\}$ must be closed.

2.9 Prove that a point cannot be simultaneously in the interior of A and on the boundary of A. (More generally, prove that A doesn't contain its boundary iff A consists of interior points.) Why doesn't this contradict our bizarre example from Lecture 3, in which we saw that [0,3) is open in $\mathbb{R}_{\geq 0}$ with respect to the Euclidean metric?

Claim. $\partial A \cap \operatorname{int} A = \emptyset$.

Proof. Observe that

 $x \in \partial A \implies \mathcal{B}_{\epsilon}(x) \cap A^{c} \neq \emptyset$ for all $\epsilon > 0 \iff \mathcal{B}_{\epsilon}(x) \not\subseteq A$ for all $\epsilon > 0 \iff x \notin \text{int } A$. \Box

Thus, we've proved that any interior point of A cannot live in ∂A . What about the rest of A? Our next result shows that all elements of A that *aren't* interior points *must* live in ∂A :

Claim. $A \setminus \text{int } A \subseteq \partial A$.

Proof. Pick any $x \in A \setminus \text{int } A$. Then $\mathcal{B}_{\delta}(x) \cap A^c \neq \emptyset$ for all $\delta > 0$. Moreover, since $x \in A$, $\mathcal{B}_{\delta}(x) \cap A \neq \emptyset$ for all $\delta > 0$. Thus $x \in \partial A$.

Our results don't contradict the example from class, since 0 isn't a boundary point of [0,3): if it were, every nonempty ball around 0 would have to intersect both [0,3) and its complement, but $\mathcal{B}_1(0) = [0,1)$ is disjoint from $[0,3)^c = [3,\infty)$.

- **2.x** (*Optional challenge problem—won't be graded*) Let $M_{n \times n}$ denote the space of all $n \times n$ matrices with real entries. Prove that $d(x, y) := \operatorname{rank}(x y)$ is a metric on $M_{n \times n}$.
- **2.y** (*Optional research project, do not submit*) In class, we played around with a visualization of our topological proof of the Fundamental Theorem of Algebra. Play around with this some more! What more insights can you glean from the picture about the polynomial or its roots? What if you change the polynomial? Are there any patterns or symmetries you notice in the images of various circles?