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MATH 374 : TOPOLOGY

Solution Set 2

2.1 Metrics on $\mathbb{R}^2$.

(a) Prove that the Euclidean metric on $\mathbb{R}^2$ is, in fact, a metric.

By definition,

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

This equals 0 iff both $(x_1 - y_1)^2, (x_2 - y_2)^2 = 0$, which happens iff $x_1 = y_1$ and $x_2 = y_2$. It's also clear that this metric is symmetric: $d(x, y) = d(y, x)$. It remains to prove the triangle inequality.

The most direct approach is quite algebraically involved. However, we can simplify this significantly by observing that the Euclidean distance is translation invariant, i.e. that $d(x, y) = d(x - z, y - z)$ for any z. After translating appropriately, we see that triangle inequality is equivalent to showing that

$$d(x, y) \leq d(x, 0) + d(0, y) \qquad (*)$$

for all x, y.

Note that $(x_1 y_2 - x_2 y_1)^2 \geq 0$. It follows that

$$(x_1 y_1 + x_2 y_2)^2 \leq (x_1^2 + x_2^2)(y_1^2 + y_2^2). \qquad (\dagger)$$

In particular,

$$-x_1 y_1 - x_2 y_2 \leq \sqrt{(x_1^2 + x_2^2)(y_1^2 + y_2^2)}.$$

From here, it's straightforward to deduce

$$\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \leq \sqrt{x_1^2 + x_2^2} + \sqrt{y_1^2 + y_2^2}$$

which is precisely $(*)$.

Discussion. Above, we saw that $(\dagger)$ implies $(*)$; it turns out that the converse implication holds as well, so the inequality $(\dagger)$ is equivalent to the triangle inequality for the Euclidean metric on $\mathbb{R}^2$. Similarly, it turns out the triangle inequality for the Euclidean metric on $\mathbb{R}^n$ is equivalent to the following:

Lemma 1 (Cauchy-Schwarz inequality)**.** *For any real numbers a_i, b_i, we have*

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

(b) Suppose $\mathcal{O}$ is a subset of $\mathbb{R}^2$ that's open with respect to the Euclidean metric. Must it also be open with respect to the taxicab metric?

Suppose $\mathcal{O} \subseteq \mathbb{R}^2$ is open with respect to the euclidean metric. Pick any $\alpha \in \mathcal{O}$; we claim that α is in the interior of $\mathcal{O}$ with respect to the taxicab metric. Since $\mathcal{O}$ is open with respect to the euclidean metric, $\exists \delta > 0$ such that the open euclidean ball of radius δ around α is entirely contained inside $\mathcal{O}$.

Consider the open taxicab ball of radius δ around α. Pick any x in this ball; by definition, the taxicab distance between x and α is smaller than δ, i.e.

$$|x_1 - \alpha_1| + |x_2 - \alpha_2| < \delta.$$

Squaring both sides, we deduce

$$|x_1 - \alpha_1|^2 + |x_2 - \alpha_2|^2 \leq |x_1 - \alpha_1|^2 + |x_2 - \alpha_2|^2 + 2|x_1 - \alpha_1| \cdot |x_2 - \alpha_2| < \delta^2.$$

This implies that x lies in the euclidean ball of radius δ around α, which we know is entirely contained in $\mathcal{O}$. We've therefore shown that every point in the taxicab ball of radius δ around α is contained entirely in $\mathcal{O}$; it follows that α is an interior point of $\mathcal{O}$ with respect to the taxicab metric, as desired.

Discussion. All this becomes obvious when looking at the pictures of open balls we drew in Lecture 3: the taxicab open ball is the largest diamond that fits inside the euclidean ball of the same radius.

(c) The Euclidean and taxicab metrics on $\mathbb{R}^2$ both have the form

$$d_p(x, y) := \left(|x_1 - y_1|^p + |x_2 - y_2|^p\right)^{1/p}$$

(d_1 is the taxicab metric, d_2 is the Euclidean metric). It turns out that d_p is a metric for every real number $p \geq 1$. (Don't worry about proving it here, although it is a fun challenge to think about when you have some spare time.) Can you describe any of the other metrics on $\mathbb{R}^2$ that we've encountered (chessboard, British Rail, and discrete) in terms of d_p?

This was very open ended, but the cleanest answers are those that describe the metric in terms of d_p *without* reference to specific inputs.

Claim. The **chessboard metric** is $d_\infty := \lim\limits_{p \to \infty} d_p$.

Proof. If $x = y$ then $d_p(x, y) = 0$ for all p, so the limit is also 0. If $x \neq y$, then without loss of generality we have

$$\max\{|x_1 - y_1|, |x_2 - y_2|\} = |x_1 - y_1| > 0.$$

Then

$$d_p(x, y) = |x_1 - y_1| \left(1 + \left(\frac{|x_2 - y_2|}{|x_1 - y_1|}\right)^p\right)^{1/p}.$$

Since this is bounded above by $|x_1 - y_1| \cdot 2^{1/p}$ and bounded below by $|x_1 - y_1|$, we see $d_p(x, y) \to |x_1 - y_1|$ as $p \to \infty$. $\square$

The **discrete metric** is $\min\{\lceil d_2 \rceil, 1\}$. (This is more artificial, of course!)

Discussion. The metric d_p is called the ℓ^p metric; you will explore it in virtually any advanced course on analysis.

2.2 In class we saw an example of a set that was open in $\mathbb{R}$, but not when viewed within $\mathbb{R}^2$. The following questions are inspired by Levi.

(a) If $A \subseteq \mathbb{R}$ (with respect to the Euclidean metric) is open, is it possible for $A \times \{1\}$ to be open in $\mathbb{R}^2$ (with respect to the Euclidean metric)? Either prove that it's never possible, or present an example where it is possible.

No, it is not possible. To see this, pick any point in $A \times \{1\}$, say, $(a, 1)$, and consider a ball $\mathcal{B}_r(a, 1)$ in $\mathbb{R}^2$. This ball clearly intersects $A \times \{1\}$, since $(a, 1) \in \mathcal{B}_r(a, 1)$. But $\mathcal{B}_r(a, 1)$ also contains the point $\left(a, 1 + \frac{r}{2}\right) \notin A \times \{1\}$. In other words, every point in $A \times \{1\}$ is on the boundary of $A \times \{1\}$, so the set cannot be closed!

(b) If $A \subseteq \mathbb{R}$ (with respect to the Euclidean metric) is closed, is $A \times \{1\}$ closed in $\mathbb{R}^2$ (with respect to the Euclidean metric)? Either prove that this is always the case, never the case, or that it is sometimes the case and sometimes not by giving explicit examples.

Yes. We'll prove this by showing that $(A \times \{1\})^c$ is open.

Pick $p \notin A \times \{1\}$. If $p_2 = 1$, then $p_1 \notin A$. Since A is closed, A^c is open, so there must exist $\delta > 0$ such that $(p_1 - \delta, p_1 + \delta) \cap A = \emptyset$. It immediately follows that $\mathcal{B}_\delta(p) \cap (A \times \{1\}) = \emptyset$. We've proved:

$$p_2 = 1 \quad \Longrightarrow \quad p \in \operatorname{int}\big((A \times \{1\})^c\big). \qquad (\clubsuit)$$

If $p \notin A \times \{1\}$ and $p_2 \neq 1$, we proceed differently. Set $\epsilon := \frac{|p_2 - 1|}{2}$, and consider $\mathcal{B}_\epsilon(p)$. For any $x \in \mathcal{B}_\epsilon(p)$, we have

$$|x_2 - p_2| \leq \sqrt{|x_1 - p_1|^2 + |x_2 - p_2|^2} < \epsilon.$$

We deduce

$$2\epsilon = |p_2 - 1| \leq |x_2 - 1| + |x_2 - p_2| < |x_2 - 1| + \epsilon,$$

whence $|x_2 - 1| > \epsilon$. In particular, $x \notin A \times \{1\}$. We conclude that $\mathcal{B}_\epsilon(p) \cap (A \times \{1\}) = \emptyset$, or in other words, that $\mathcal{B}_\epsilon(p) \subseteq (A \times \{1\})^c$. We've proved:

$$p_2 \neq 1 \quad \Longrightarrow \quad p \in \operatorname{int}\big((A \times \{1\})^c\big). \qquad (\heartsuit)$$

Combining ($\clubsuit$) and ($\heartsuit$) shows that $(A \times \{1\})^c$ consists entirely of interior points, hence must be open; it follows that $A \times \{1\}$ must be closed.

2.3 Let $\mathbb{R}^\infty$ be the set of all sequences of real numbers. Determine (with proof) whether each of the following is a metric on $\mathbb{R}^\infty$.

(a) $d(x, y) := \max\{|x_n - y_n|\}$.

This isn't even well-defined: the sequences $x_n = 0$ and $y_n = n$ would be infinitely far apart by this metric, which isn't allowed.

(b) $d(x,y) := \begin{cases} \frac{1}{n+1} & \text{if } \exists n \geq 0 \text{ s.t. } x_i = y_i \text{ for all } i \leq n \text{ and } x_{n+1} \neq y_{n+1} \\ 0 & \text{if } x = y. \end{cases}$

Yes, this is a metric. The first two properties are straightforward to verify, so we focus on the triangle inequality $d(x,z) + d(y,z) \geq d(x,y)$. If $x = y$, this is obvious from the definition of d, since $d(x,z) \geq 0$ and $d(y,z) \geq 0$ while $d(x,y) = 0$. Thus it suffices to consider the case when $d(x,y) = 1/(n+1)$ for some non-negative integer n, i.e. when x and y agree for the first n terms but disagree at the $(n+1)$st. There are two possibilities for a third sequence z: either it disagrees with x before the nth term, or it agrees with x up until the nth term (and possibly beyond).

Suppose $d(x,z) = \frac{1}{m+1}$ for some $m < n$. Since $d(\alpha, \beta) \geq 0$ for all α, β, we have

$$d(x,z) + d(z,y) \geq d(x,z) = \frac{1}{m+1} > \frac{1}{n+1} = d(x,y),$$

so triangle inequality holds in this case.

If instead z agrees with x through term n (and possibly beyond), then z must also agree with y at least through term n. However, it cannot agree with both x and y at the $(n+1)$st term, since $x_{n+1} \neq y_{n+1}$ by hypothesis. Without loss of generality, assume $z_{n+1} \neq x_{n+1}$. It follows that $d(x,z) = 1/(n+1)$, which implies

$$d(x,z) + d(z,y) \geq d(x,z) = \frac{1}{n+1} = d(x,y).$$

(c) $d(x,y) := \sum_{n=1}^{\infty} \frac{1}{2^n}\big(1 - \delta(x_n, y_n)\big)$ where $\delta(x,y) = 1$ if $x = y$ and 0 otherwise.

Let D denote the discrete metric on $\mathbb{R}$. Observe that

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} D(x_n, y_n).$$

The sum is zero iff each term is zero (which happens iff the sequences agree), and the definition is symmetric with respect to the sequences x, y. The triangle inequality is inherited term-wise from the discrete metric.

2.4 Given finite sets $A, B \subseteq \mathbb{Z}$, let $A - B := \{a - b : a \in A, b \in B\}$ and define the *Ruzsa distance* by

$$d(A,B) := \log \frac{|A - B|}{\sqrt{|A|}\sqrt{|B|}}.$$

(Here $|S|$ denotes the number of elements in a set S.) Is this a metric?

No, this is not a metric, because $d(A,A)$ might be nonzero. For example, let $A := \{2,3\}$. Then $A - A = \{0, \pm 1\}$, so $d(A,A) = \log \frac{3}{2} \neq 0$.

Discussion. Amazingly, the Ruzsa distance does satisfy the triangle inequality—a fun exercise!

2.5 Given a metric space (X, d) where X has at least 2 elements. Prove that there exists a metric on X that's neither the discrete metric nor equal to the metric d.

Observe that rescaling a metric doesn't change that it's a metric. Also, summing any two metrics produces a metric. Thus, any linear combination $D(x,y) := \alpha d_1(x,y) + \beta d_2(x,y)$ of any two metrics d_1, d_2 is a metric as well.

We can form a new metric in other ways, too. Here's a nice example: given a metric d, set $D(x,y) := \frac{d(x,y)}{1+d(x,y)}$. I claim that D is a metric. It's easy to verify the first two properties, so it suffices to handle the triangle inequality:

$$\begin{aligned} D(x,z) = 1 - \frac{1}{1+d(x,z)} &\leq 1 - \frac{1}{1+d(x,y)+d(y,z)} = \frac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)} \\ &= \frac{d(x,y)}{1+d(x,y)+d(y,z)} + \frac{d(y,z)}{1+d(x,y)+d(y,z)} \\ &\leq \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)} = D(x,y) + D(y,z). \end{aligned}$$

It's an exercise to check that $D \neq d$ and also cannot equal the discrete metric.

2.6 Prove that a point cannot be simultaneously in the interior of A and on the boundary of A. Why doesn't this contradict our bizarre example from Lecture 3, in which we saw that $[0,3)$ is open in $\mathbb{R}_{\geq 0}$ with respect to the Euclidean metric?

Claim. $\partial A \cap \mathrm{int} A = \emptyset$.
Proof. Observe that

$$x \in \partial A \implies \mathcal{B}_\epsilon(x) \cap A^c \neq \emptyset \text{ for all } \epsilon > 0 \iff \mathcal{B}_\epsilon(x) \not\subseteq A \text{ for all } \epsilon > 0 \iff x \notin \mathrm{int}\, A.$$

This concludes the proof. □

Thus, we've proved that any interior point of A cannot live in ∂A. What about the rest of A? Our next result shows that all elements of A that *aren't* interior points *must* live in ∂A:

Claim. $A \setminus \mathrm{int}\, A \subseteq \partial A$.
Proof. Pick any $x \in A \setminus \mathrm{int}\, A$. Then $\mathcal{B}_\delta(x) \cap A^c \neq \emptyset$ for all $\delta > 0$. Moreover, since $x \in A$, $\mathcal{B}_\delta(x) \cap A \neq \emptyset$ for all $\delta > 0$. Thus $x \in \partial A$. □

Our results don't contradict the example from class, since 0 isn't a boundary point of $[0,3)$: if it were, every nonempty ball around 0 would have to intersect both $[0,3)$ and its complement, but $\mathcal{B}_1(0) = [0,1)$ is disjoint from $[0,3)^c = [3,\infty)$.

2.7 Recall our graph theoretic example of a metric space: $\{A, B, C, D\}$ with the distance between any two of A, B, C being 2 and the distance between D and any one of A, B, C being 1.

(a) What's $\partial\{B\}$?

$\partial\{B\} = \emptyset$. To see this, first observe that B cannot be in the boundary, since the ball of radius $1/2$ around B only contains B. Similarly, a ball of radius $1/2$ around any other point would not contain B. Thus, there's no point that can belong to the boundary of $\{B\}$.

(b) Describe all the open sets in this space.

Every subset of $\{A, B, C, D\}$ is open. Indeed, for any $p \in \{A, B, C, D\}$, the ball of radius $1/2$ around p only contains p, so every choice of p is an interior point in whichever set it belongs to.

(c) Describe all the closed sets in this space.

Since all subsets of the space are open, their complements must all be closed—or in other words, every subset of the space is clopen.

2.8 Recall from class that $\mathcal{B}_r(p)$ denotes the "open ball" of radius r around the point p.

(a) Prove that for any $p \in X$ and any positive r, $\mathcal{B}_r(p)$ is open.

Pick any $q \in \mathcal{B}_r(p)$, and set $\epsilon := r - d(p, q)$.

Claim. $\mathcal{B}_\epsilon(q) \subseteq \mathcal{B}_r(p)$

Proof. Pick $x \in \mathcal{B}_\epsilon(q)$. Then

$$d(x, p) \leq d(x, q) + d(p, q) < \epsilon + d(p, q) = r. \qquad \square$$

Thus, we've shown that every point of $\mathcal{B}_r(p)$ is interior; it follows that $\mathcal{B}_r(p)$ is open.

(b) What if $r = 0$ above? Is $\mathcal{B}_0(p)$ open? You must either prove that it's always open, prove that it's never open, or provide examples to show that it can be sometimes open and sometimes not open.

$\mathcal{B}_0(p) = \emptyset$, an open set.