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MATH 374 : TOPOLOGY

Solution Set 3

3.1 Consider the space $X := \{f : [0,1] \to \mathbb{R}, \text{ a continuous function}\}$.

(a) Explain (with examples) why $\Delta(f,g) := \max_{t \in (0,1)} |f(t) - g(t)|$ is not a metric on X.

 $\Delta(f,g)$ satisfies all three properties of a metric, but it fails to satisfy something more fundamental: it's not a function from $X \times X \to \mathbb{R}$. For example, if f(t) = t and g(t) = 0 for all $t \in [0, 1]$, then

$$\Delta(f,g) = \max_{t \in (0,1)} |t| = \max(0,1)$$

Since the right hand side doesn't exist, $\Delta(f,g)$ isn't defined for this choice of f and g.

(b) Prove that $d(f,g) := \max_{t \in [0,1]} |f(t) - g(t)|$ is a metric on X.

This time, the proposed metric is well-defined (thanks to the Extreme Value Theorem). The only metric property that's not straightforward is the triangle inequality. To see that it holds, suppose $f, g, h \in X$, and pick any $a \in [0, 1]$. Then

$$|f(a) - h(a)| \le |f(a) - g(a)| + |g(a) - h(a)| \le d(f,g) + d(g,h).$$

Since the right hand side is independent of a, we deduce that

$$d(f,h) = \max_{a \in [0,1]} |f(a) - h(a)| \le d(f,g) + d(g,h).$$

(c) Prove that any function in X is completely determined by its behavior on \mathbb{Q} . In other words, show that if $f, g \in X$ and f(q) = g(q) for all $q \in \mathbb{Q}$, then f = g.

It suffices to prove that if $h \in X$ vanishes at all rational inputs, then h vanishes everywhere. Pick any $a \in [0, 1]$. For any $\epsilon > 0$ there exists some neighborhood of a on which

$$|h(x) - h(a)| < \epsilon.$$

Now observe that h(x) = 0 somewhere in this neighborhood, since the rationals are dense in \mathbb{R} . It instantly follows that

 $|h(a)| < \epsilon$

for any $\epsilon > 0$. We conclude that h(a) = 0, as claimed.

3.2 Given a function $f: X \to Y$, where X and Y are metric spaces. We proved in class that f is continuous on X if and only if $f^{-1}(\mathcal{B})$ is open in X for every open ball \mathcal{B} in Y. Use this to prove the following

Theorem. f is continuous on X if and only if $f^{-1}(\mathcal{O})$ is open in X for every open set \mathcal{O} in Y.

(In words: a function is continuous iff the preimage of any open set is open.)

(\Leftarrow) If the preimage of any open set is open, then in particular, the preimage of any open ball is open. It follows from our work in class that f must be continuous.

 (\Rightarrow) Suppose f is continuous, and pick any open set \mathcal{O} in Y. For every point $y \in \mathcal{O}$, there exists an open ball \mathcal{A}_y such that $y \in \mathcal{A}_y \subseteq \mathcal{O}$. We have

$$f^{-1}(\mathcal{O}) = f^{-1}\left(\bigcup_{y\in\mathcal{O}}\mathcal{A}_y\right) = \bigcup_{y\in\mathcal{O}}f^{-1}(\mathcal{A}_y).$$

This is a union of preimages of open balls, which (from class) we know are all open. We deduce that $f^{-1}(\mathcal{O})$ is open, as claimed.

3.3 In class we saw that any collection of disjoint open intervals must be countable. Does this also hold for closed intervals?

No! For example, $\mathbb{R} = \bigsqcup_{x \in \mathbb{R}} [x, x]$.

3.4 In class we described the Cantor set and some of its properties. Here we explore this topic more carefully. First, we recall the construction of the Cantor set. (This was done in class using less formal language). We begin with the open interval $\mathcal{O}_1 := (1/3, 2/3)$. Next, for each $n \ge 1$ define

$$\mathcal{O}_{n+1} := \left(\frac{1}{3} \cdot \mathcal{O}_n\right) \cup \left(\frac{2}{3} + \frac{1}{3} \cdot \mathcal{O}_n\right),$$

where $\alpha \cdot X := \{\alpha x : x \in X\}$ and $\beta + Y := \{\beta + y : y \in Y\}$. Finally, set

$$\mathcal{C} := [0,1] \setminus \left(\bigcup_{n=1}^{\infty} \mathcal{O}_n \right).$$

It immediately follows that C is closed and bounded, hence that C is *compact*.

(a) Prove that \mathcal{C} has empty interior, i.e. that no points of \mathcal{C} are interior points.

Let

$$\mathcal{C}_m := [0,1] \setminus \big(\bigcup_{n \le m} \mathcal{O}_n\big);$$

by definition of the Cantor set, $C_m \supseteq C$ for every N. Note that (by induction) C_m is the disjoint union of 2^m closed intervals, each of length $1/3^m$.

Pick any point $x \in int(\mathcal{C})$; by definition, there exists $\epsilon > 0$ such that $\mathcal{B}_{\epsilon}(x) \subseteq \mathcal{C}$, whence

 $\mathcal{B}_{\epsilon}(x) \subseteq \mathcal{C}_m$

for every *m*. But for sufficiently large *m* we have $\frac{1}{3^m} < \epsilon$, so \mathcal{C}_m cannot contain any interval of length ϵ ! We conclude that the interior of \mathcal{C} must be empty.

(b) Prove that \mathcal{C} has no isolated points.

We continue using the notation C_m defined in the previous solution. Recall that C_m is the disjoint union of 2^m closed intervals, each of length $\frac{1}{3^m}$; moreover, observe that the endpoint of any one of these closed intervals must live in C. This implies that any point of C_m is within a distance of $\frac{1}{3^m}$ of some point of C. In particular, for any $p \in C$ and any m, we have that p is within a distance of $\frac{1}{3^m}$ of some other point of C. Since $\frac{1}{3^m}$ can be made arbitrarily small, p cannot be isolated.

all the \mathcal{O}_n 's are disjoint, the total length is $\sum_{n=1}^{\infty} \frac{1}{3} \left(\frac{2}{3}\right)^{n-1} = \frac{1/3}{1-2/3} = 1.$

(d) Prove that $x \in \mathcal{C}$ iff x has a ternary (i.e. base 3) expansion that doesn't use the digit 1 anywhere.

FIRST DESCRIPTION. The first set we remove, \mathcal{O}_1 , consists of all numbers with ternary expansion of the form $0.1\cdots$. The next set, \mathcal{O}_2 , consists of remaining numbers whose second ternary digit is a 1. Similarly, \mathcal{O}_n consists of all numbers between 0 and 1 such that the first n-1 ternary digits are exclusively 0 and 2, and the n^{th} ternary digit is 1. It follows that any $x \notin \bigcup_{n\geq 1} \mathcal{O}_n$ has a ternary expansion that uses only 0s and 2s.

SECOND DESCRIPTION. Above we defined C_m to be the m^{th} stage of forming the Cantor set, where we have created 2^m disjoint closed intervals each of length $1/3^m$. Here we develop a convenient nomenclature for the individual closed intervals composing C_m . We will write

$$\mathcal{C}_m = \bigsqcup_{\ell = m ext{-digit binary number}} I_\ell.$$

Thus

$$C_1 = I_0 \sqcup I_1$$

$$C_2 = I_{00} \sqcup I_{01} \sqcup I_{10} \sqcup I_{11}$$

:

For any closed interval I, let $\alpha(I)$ denote the left endpoint of I and $\beta(I)$ denote the right endpoint, i.e. $I = [\alpha(I), \beta(I)]$. We will now define I_{ℓ} recursively, as follows.

First, set $I_0 := [0, 1/3]$ and $I_1 := [2/3, 1]$. Next, given an (m-1)-digit binary number ℓ , we will define $I_{\ell 0}$ and $I_{\ell 1}$ in terms of the endpoints of the interval I_{ℓ} :

$$I_{\ell 0} := [\alpha(I_{\ell}), \alpha(I_{\ell}) + \frac{1}{3^{m}}]$$
$$I_{\ell 1} := [\beta(I_{\ell}) - \frac{1}{3^{m}}, \beta(I_{\ell})]$$

A straightforward induction proves our assertion that C_m is the disjoint union of the closed intervals I_{ℓ} over all *m*-digit binary numbers ℓ .

Finally, observe that any $x \in C_m$ must live in an interval of the form $I_{d_1d_2\cdots d_m}$ with each $d_i = 0$ or 1. A final proof by induction shows that

$$x \in I_{d_1 d_2 \cdots d_m} \quad \iff \quad x = 0.e_1 e_2 \cdots e_m \dots \text{ in ternary,}$$

where $e_i := 2d_i$; in particular, the first *m* ternary digits of *x* must be 0 or 2. Since $x \in C$ requires that $x \in C_m$ for every *m*, we deduce the claim.

(e) Given sets \mathcal{A} and \mathcal{B} of real numbers, define their sum and difference to be

$$\mathcal{A} + \mathcal{B} := \{ a + b : a \in \mathcal{A}, b \in \mathcal{B} \} \qquad \qquad \mathcal{A} - \mathcal{B} := \{ a - b : a \in \mathcal{A}, b \in \mathcal{B} \}.$$

Prove that $\mathcal{C} + \mathcal{C} = [0, 2]$ and $\mathcal{C} - \mathcal{C} = [-1, 1]$.

Perhaps the easiest approach is to start by proving

$$\frac{1}{2}\mathcal{C} + \frac{1}{2}\mathcal{C} = [0, 1]. \tag{1}$$

The \subseteq containment is obvious. To prove the other direction, pick any $x \in [0, 1]$ and write its ternary expansion as

 $x = 0.a_1a_2a_3\cdots$

We can easily write x as a sum of two ternary numbers $0.b_1b_2b_3\cdots$ and $0.c_1c_2c_3\cdots$, all of whose digits are 0 or 1: if $a_k = 0$, set $b_k = c_k = 0$; if $a_k = 1$, set $b_k = 0$ and $c_k = 1$; if $a_k = 2$, set $b_k = c_k = 1$.

From (1), it's immediate that C + C = [0, 2]. To deduce the second claim, observe that -C = C - 1, whence

$$\mathcal{C} - \mathcal{C} = \mathcal{C} + \mathcal{C} - 1 = [-1, 1].$$

- **3.5** This problem builds on the previous one and introduces the notorious **Cantor-Lebesgue function**. This is a function $F : [0,1] \rightarrow [0,1]$ with the seemingly paradoxical properties that
 - F is continuous everywhere on [0, 1].
 - F(0) = 0 and F(1) = 1.
 - The measure (i.e. total length) of the set $\{x \in [0,1] : F'(x) = 0\}$ equals 1!
 - (a) Consider $f: \mathcal{C} \to [0, 1]$ defined by

$$f(x) := \sum_{k=1}^{\infty} \frac{a_k/2}{2^k},$$

where $x = 0.a_1a_2a_3...$ is a ternary expansion of x that doesn't use the digit 1. Prove that f is well-defined and continuous on C, and that f(0) = 0 and f(1) = 1.

To prove continuity means to prove that nearby inputs produce nearby outputs. If two inputs are nearby, that means their ternary expansions agree for the first bunch of digits. But this means the corresponding outputs have binary expansions that agree for the first bunch of digits, hence are close to each other.

(b) Prove that f is surjective. [Note that, bizarrely, f maps a measure 0 set onto a set of measure 1.]

Pick any element $\alpha \in [0, 1]$ and write it in binary. Now double each digit and interpret the number in ternary! The image of this number under f is α .

(c) Prove that if $a, b \in \mathcal{C}$ and $(a, b) \subset [0, 1] \setminus \mathcal{C}$, then f(a) = f(b).

The way the problem is set up, a is the right endpoint of an interval removed from C_k , while b is the left endpoint of the next interval over in C_k . A bit of thought shows that the ternary expansions of a and b are the same, except that the 3^{-k} -th digit of a is a 1 while the 3^{-k} -th digit of b is a 2. In other words, we can write

$$a = 0.d_1 d_2 \cdots d_{k-1} 0\overline{2}$$
 and $b = 0.d_1 d_2 \cdots d_{k-1} 2$.

Applying f to both these yields binary numbers

$$f(a) = 0.e_1e_2\cdots e_{k-1}0\overline{1} = 0.e_1e_2\cdots e_{k-1}1 = f(b).$$

(d) Deduce the existence of the Cantor-Lebesgue function.

First, we extend f above to a function $F:[0,1]\to [0,1]$ by setting

$$F(x) = \begin{cases} f(x) & \text{if } x \in \mathcal{C} \\ f(a) & \text{if } x \in (a,b) \setminus \mathcal{C} \text{ with } a, b \in \mathcal{C}. \end{cases}$$

We've already verified that F is continuous everywhere in C, and it's constant everywhere else so it is clearly continuous outside of C as well. It's clear that f(0) = 0 and f(1) = 1, which implies the same is true of F. Finally, F is constant everywhere on $[0,1] \setminus C$, and we proved above that this set has measure 1, so F is differentiable on a set of measure 1!