BASIS OF A TOPOLOGY

LEO GOLDMAKHER

1. Motivation

Recall that in $\mathbb{R}_{\text{usual}}$ a set $\mathcal{A}$ is open iff every point in that set is interior (i.e. iff for each $\alpha \in \mathcal{A}$ we have $\mathcal{B}_\delta(\alpha) \subseteq \mathcal{A}$ for some $\delta > 0$). But there's a more explicit description: $\mathcal{A}$ is open iff it's a countable union of bounded open intervals. (Note that the empty set is produced this way: it's the "empty" union.) What makes this particularly nice is that we're starting with a conceptually simple collection of sets—bounded open intervals—and then only using the union operation to generate all possible open sets. At the end of the day, the open sets are closed under finite intersection as well, but the fact that we don't use intersections in the process of generating all opens out of the open intervals makes the topology of $\mathbb{R}_{\text{usual}}$ much easier to imagine!

Can we do the same thing in a more general topological space? In other words, given a topology on X, can we come up with collection of particularly nice sets that, when union-ed together in all possible ways, generate the topology? It's not at all obvious how to accomplish this, so we ask an easier question: given a set X and a bunch of subsets of X, say, $\mathcal{B}$. If we take all possible unions of all the elements of $\mathcal{B}$, does this generate a topology on X? Not necessarily! For example, $\{(0,1)\}$ doesn't generate a topology on $\mathbb{R}$, because it doesn't generate the set $\mathbb{R}$ itself, which *must* be in the topology. More generally, we see that for $\mathcal{B}$ to have a hope of generating a topology on X, it must *cover* X: the union of all elements of $\mathcal{B}$ must contain X. (In fact, this forces the union to equal X.)

Is this enough to guarantee that $\mathcal{B}$ generates a topology? Again, not necessarily: a topology is closed under finite unions, whereas the set generated by all the sets in $\mathcal{B}$ might not be! Thus, in addition to covering X we must require that finite intersections of elements of $\mathcal{B}$ are in the topology generated by $\mathcal{B}$, i.e. are unions of elements of $\mathcal{B}$. It turns out that these necessary conditions are also sufficient for $\mathcal{B}$ to generate a topology! This motivates the following definition:

Definition. A *basis* on a set X is any set $\mathcal{B} \subseteq \mathcal{P}(X)$ such that

(*i*) $\displaystyle\bigcup_{\mathcal{A}\in\mathcal{B}} \mathcal{A} = X$, and

(*ii*) for any $S, T \in \mathcal{B}$, the set $S \cap T$ is a union of elements of $\mathcal{B}$.

Of course, this is nice, but does it really do what we want? That's the subject of the next section.

2. Justification

The purpose of a basis is to generate a topology out of a smaller (and hopefully simpler) set. But does this actually happen?

Proposition 2.1. *If $\mathcal{B}$ is a basis on X, then the collection of all possible unions of elements of $\mathcal{B}$ is a topology on X.*

Proof. Let $\mathcal{T}$ denote the collection of all possible unions of elements of $\mathcal{B}$. To verify that $\mathcal{T}$ is a topology, we must check

(1) $\emptyset, X \in \mathcal{T}$,
(2) $\mathcal{T}$ is closed under unions, and
(3) $\mathcal{T}$ is closed under finite intersections.

Claim (1) is a straightforward consequence of property (i) of a basis: $\emptyset \in \mathcal{T}$ because we can take the empty union of elements from $\mathcal{B}$, and $X \in \mathcal{T}$ because X is covered by $\mathcal{B}$. Claim (2) is true because every element of $\mathcal{T}$ is a union of elements of $\mathcal{B}$, so a union of elements of $\mathcal{T}$ is also a union of elements of $\mathcal{B}$. This leaves claim (3), which is a bit trickier.

To warm up, let's consider the intersection of two elements of $\mathcal{T}$. If $U, V \in \mathcal{T}$, then (by definition of $\mathcal{T}$) we know that U is the union of a bunch of sets in $\mathcal{B}$, as is V:

$$U = \bigcup_{\alpha \in I} B_\alpha \qquad \text{and} \qquad V = \bigcup_{\alpha \in I} C_\alpha$$

where all the B_α's and C_β's are elements of $\mathcal{B} \cup \{\emptyset\}$. Note that we're using the same index set I in the descriptions of U and V, which seems like an unjustified assumption. We can get away with this by taking all the actual B_α's we need to express U (with the α ranging over just some subset of I) and then letting the remaining B_α's all be the empty set; then we do the same thing for the C_α's. It follows that

$$U \cap V = \left(\bigcup_{\alpha \in I} B_\alpha \right) \cap \left(\bigcup_{\alpha \in I} C_\alpha \right) = \bigcup_{\alpha, \beta \in I} \left(B_\alpha \cap C_\beta \right).$$

But now property (ii) of bases implies that each $B_\alpha \cap C_\beta$ is a union of basis elements, so the union over all these is also a union of basis elements, proving (3) in the special case of an intersection of two elements in $\mathcal{T}$. The extension to arbitrary finite intersection is a straightforward exercise in induction. $\square$