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NAME: ________________________________

Williams College
Department of Mathematics and Statistics

MATH 394 : GALOIS THEORY

Problem Set 1 – due Thursday, February 8th

INSTRUCTIONS:
This assignment must be turned in to my mailbox (on the right as you enter Bascom) by **4pm** sharp. Assignments may be submitted later than this by email to Alyssa, but no later than 4pm on Friday; in this case, the grade will be reduced by 10%.
Assignments submitted later than Friday at 4pm will not be graded.

Please print and attach this page as the first page of your submitted problem set.

PROBLEM	GRADE
1.1	
1.2	
1.3	
1.4	
Total	

Please read the following statement and sign below:

I understand that I am not allowed to use the internet to assist with this assignment. I also understand that I must write down the final version of my assignment in isolation from any other person. I pledge to abide by the Williams honor code.

SIGNATURE:____

Problem Set 1

1.1 In class we observed that complex conjugation 'commutes' with addition and multiplication, in the sense that $\overline{x+y} = \overline{x} + \overline{y}$ and $\overline{xy} = \overline{x} \cdot \overline{y}$ for any $x, y \in \mathbb{C}$.

(a) Prove that complex conjugation commutes with all four operations $+, -, \times, \div$. (We asserted it for the first two operations, but didn't prove it for any of them.)

(b) Prove that complex conjugation commutes with the functions exp() and sin(). [*Hint: how can one define these functions meaningfully for complex inputs? Taylor series! Don't stress out too much about convergence issues.*]

(c) Can you construct a function f and a choice of $z \in \mathbb{C}$ such that $f(z)$ is defined, but $\overline{f(z)} \neq f(\overline{z})$?

1.2 Show that for any $a, b \in \mathbb{Q}$ such that $\sqrt{b} \notin \mathbb{Q}$, the two numbers $a \pm \sqrt{b}$ are algebraically indistinguishable over $\mathbb{Q}$. [*Hint: start by proving that $\pm\sqrt{b}$ are algebraically indistinguishable.*]

1.3 Prove that complex conjugates are algebraically indistinguishable over $\mathbb{R}$.

1.4 The goal of this problem is to prove the following assertion from lecture:

Claim. *The only choice of rational numbers a, b, c satisfying $a + b\sqrt[3]{2} + c(\sqrt[3]{2})^2 = 0$ is $a = b = c = 0$. (In other words, $1, \sqrt[3]{2}, (\sqrt[3]{2})^2$ are linearly independent over $\mathbb{Q}$.)*

Define
$$S := \{(a, b, c) \in \mathbb{Z}^3 : a + b\sqrt[3]{2} + c(\sqrt[3]{2})^2 = 0\}.$$

We call the element $(0, 0, 0)$ the *trivial* element of S.

(a) Prove that S forms a group under addition.

(b) Prove that if $(x, y, z) \in S$ is nontrivial, then $xyz \neq 0$.

(c) We call $(a, b, c) \in \mathbb{Z}^3$ *primitive* iff $\gcd(a, b, c) = 1$. Prove that if S contains a nontrivial element, then it must contain a primitive nontrivial element.

(d) Prove that S only contains the trivial element. [*Hint: recall that given any $a, b, c \in \mathbb{Z}$, there exist $x, y, z \in \mathbb{Z}$ such that $ax + by + cz = \gcd(a, b, c)$. You may use this statement without proof.*]

(e) Prove the claim. (*Careful! There's something to check here.*)