Instructor: Leo Goldmakher

Name: ______________________________

Williams College
Department of Mathematics and Statistics

MATH 394 : GALOIS THEORY

Problem Set 2 – due Thursday, February 15th

INSTRUCTIONS:

This assignment must be turned in to my mailbox (on the right as you enter Bascom) by **4pm** sharp. Assignments may be submitted later than this by email to Alyssa, but no later than 4pm on Friday; in this case, the grade will be reduced by 10%.

Assignments submitted later than Friday at 4pm will not be graded.

Please print and attach this page as the first page of your submitted problem set.

PROBLEM	**GRADE**
2.1	
2.2	
2.3	
2.4	
Total	

Please read the following statement and sign below:

I understand that I am not allowed to use the internet to assist with this assignment. I also understand that I must write down the final version of my assignment in isolation from any other person. I pledge to abide by the Williams honor code.

SIGNATURE:______________________________

Problem Set 2

2.1 (Hi, this is abstract algebra. Remember me?) Given a group G and two subsets $A, B \subseteq G$, we define the *commutator of* A *and* B, denoted $[A, B]$, to be the group *generated* by all the elements $\{[a, b] : a \in A, b \in B\}$, where $[a, b] := aba^{-1}b^{-1}$. This problem does not assume prior experience with these objects, so **please do not look up any information on commutators.** However, you may look up the basic notions from group theory: subgroup, normal subgroup, quotient group, isomorphism, even and odd permutations.

(a) Prove that for any permutations $\sigma, \tau \in S_n$, the permutation $[\sigma, \tau]$ is even.

(b) Show that $[S_3, S_3] \simeq \mathbb{Z}_3$, and that $[\mathbb{Z}_3, \mathbb{Z}_3]$ is trivial. [*Hint: If you use part (a) you get to be lazy!*]

(c) Given a finite group G, consider the set $\{N \trianglelefteq G : G/N \text{ is abelian}\}$. (Recall that $N \trianglelefteq G$ means that N is a normal subgroup of G.) Prove that the smallest element of this set is $[G, G]$.

2.2 The following questions refer to the write-up on Arnold's theorem.

(a) Solve Exercise 1 (from the write-up) in the special case $n = 1$, $\alpha = 1/3$, and $f(x) = x$ for all $x \in \mathbb{C}$. (Note that $\mathcal{F}_1 = \mathbb{C}$.) [**Extra Credit:** Solve the exercise for a general continuous function f.]

(b) Exercise 4 from the write-up.

(c) In view of Arnold's proof, explain what problem **2.1**(b) implies about the cubic formula.

2.3 In class, we saw that given one root of a cubic polynomial, we can use the quadratic formula to express the other two roots. The goal of this exercise is to generalize this principle.

(a) Carry out the above for a general (monic) cubic. In other words, given $f(x) = x^3 + a_2x^2 + a_1x + a_0$, and β such that $f(\beta) = 0$, determine the other two roots of $f(x)$ in terms of β (and the a_i's).

(b) Now suppose you're given the polynomial $f(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, and you happen to know that $f(\beta) = 0$. Write down a cubic polynomial whose roots are precisely the other three roots of $f(x)$.

(c) Generalize part (b) to arbitrary monic polynomials of degree n.

2.4 In class we discovered how to find the roots of a specific cubic. Here we explore this further.

(a) Using the method we described in class, derive a formula which always produces a root of $x^3 + cx + d$. [*You should arrive at* $\sqrt[3]{-\frac{d}{2} + \sqrt{\frac{c^3}{27} + \frac{d^2}{4}}} + \sqrt[3]{-\frac{d}{2} - \sqrt{\frac{c^3}{27} + \frac{d^2}{4}}}$]

(b) Note that 3 is a root of $x^3 - 3x - 18$. What does the formula from part (a) give? Can you show that this is equal to 3?

(c) Consider the cubic $(x - 1)(x - 2)(x + 3)$. Which root does the formula from part (a) produce? (This type of phenomenon is what forced people to recognize the existence – or at least, the utility! – of imaginary numbers.)

(d) Explicitly determine one solution of the equation $x^3 - 6x^2 + 21x - 22 = 0$.