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Williams College
Department of Mathematics and Statistics

MATH 394 : GALOIS THEORY

Problem Set 4 – due Thursday, March 1st

INSTRUCTIONS:
This assignment must be turned in to my mailbox (on the right as you enter Bascom) by **4pm** sharp. Assignments may be submitted later than this by email to Alyssa, but no later than 4pm on Friday; in this case, the grade will be reduced by 5%.
Assignments submitted later than Friday at 4pm will not be graded.

Please print and attach this page as the first page of your submitted problem set.

PROBLEM	**GRADE**
4.1	
4.2	
4.3	
4.4	
4.5	
4.6	
4.7	
Total	

Please read the following statement and sign below:

I understand that I am not allowed to use the internet to assist with this assignment. I also understand that I must write down the final version of my assignment in isolation from any other person. I pledge to abide by the Williams honor code.

SIGNATURE:____

Problem Set 4

4.1 Let K be a field.

(a) Prove that $0x = 0$ for all $x \in K$, and that $xy = 0$ implies $x = 0$ or $y = 0$.

(b) Prove that char K must either be 0 or prime.

(c) Given two fields K and K', prove that if char $K \neq$ char K' then there's no embedding of K into K'.

(d) Give an example of two non-isomorphic fields which have the same characteristic.

(e) Suppose L/K is a field extension. Prove that char $K =$ char L.

4.2 Given a field K, define P_K to be the intersection of all subfields of K.

(a) Prove that P_K is a field.

(b) If char $K = 0$, then P_K is isomorphic to a familiar field. Which one? Prove it.

(c) If char $K = p$, then P_K is isomorphic to a familiar field. Which one? Prove it.

4.3 Let $\omega := e^{2\pi i/3}$. Show that $\mathbb{Q}(\omega) = \mathbb{Q}[\omega]$. (Recall that $\mathbb{Q}(\alpha)$ denotes the subfield of $\mathbb{C}$ generated by α, whereas $\mathbb{Q}[\alpha]$ denotes the set of all polynomials in α with rational coefficients.)

4.4 Fun with quotients!

(a) Prove that $\mathbb{Q}[t]/(t^3 - 2) \simeq \mathbb{Q}[\sqrt[3]{2}]$.

(b) Prove that $\mathbb{Q}[\sqrt[3]{2}] = \mathbb{Q}(\sqrt[3]{2})$. Do not use algebraic number theory! [*Hint: you may find the identity* $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$ *useful.*]

(c) Does there exist any $\alpha \in \mathbb{C}$ such that $\mathbb{Q}[t]/(t^3 - 2) \simeq \mathbb{Q}(\alpha)$ but $\mathbb{Q}(\alpha) \neq \mathbb{Q}(\sqrt[3]{2})$? Prove.

(d) Are the two fields $\mathbb{Q}[t]/(t^2 + 3)$ and $\mathbb{Q}[t]/(t^2 + 1)$ isomorphic? Why or why not? Prove.

(e) Are the two fields $\mathbb{R}[t]/(t^2 + 3)$ and $\mathbb{R}[t]/(t^2 + 1)$ isomorphic? Why or why not? Prove.

4.5 The purpose of this is to become familiar with the rational root test.

(a) Given $f \in \mathbb{Z}[x]$, say,

$$f(x) = a_n x^n + \cdots + a_1 x + a_0.$$

Prove that any rational root of f can be written $\pm\frac{k}{\ell}$ with $k \mid a_0$ and $\ell \mid a_n$.

(b) Use (a) to determine whether or not $x^3 + x^2 - 5x + 2$ is reducible over $\mathbb{Q}$.

(c) Use (a) to determine whether or not $3x^3 + x^2 - 5x + 2$ is reducible over $\mathbb{Q}$.

4.6 Recall from class that $f \in \mathbb{Z}[t]$ is *primitive* iff the coefficients of f are relatively prime. Prove that the product of two primitive polynomials is primitive.

4.7 The goal of this problem is to introduce a new irreducibility test.

(a) Prove that $|f^{-1}(k)| \leq \deg f$ for any nonconstant $f \in \mathbb{Z}[t]$. [Here $f^{-1}(k) := \{n \in \mathbb{Z} : f(n) = k\}$.]

(b) Given $f \in \mathbb{Z}[t]$, consider the set

$$P_f := \{n \in \mathbb{Z} : |f(n)| = 1 \text{ or prime}\}.$$

Suppose f is monic and non-constant. Prove that if $|P_f| \geq 2\deg(f) + 1$ then f is irreducible over $\mathbb{Q}$. [Colloquially this says that if f outputs primes at a lot of inputs, it must be irreducible. A famous open conjecture (due to Bunyakovsky) asserts a strong converse: if $f \in \mathbb{Z}[t]$ is primitive and irreducible then it outputs infinitely many primes. This has been proved for all primitive degree 1 polynomials, but is not known to hold for any single example with $\deg f \geq 2$.]

(c) Use the above to prove that $x^4 - 2x^3 + 9x - 1$ is irreducible over $\mathbb{Q}$.