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NAME: ______________________________

Williams College
Department of Mathematics and Statistics

MATH 394 : GALOIS THEORY

Problem Set 5 – due Thursday, March 8th

INSTRUCTIONS:
This assignment must be turned in to my mailbox (on the right as you enter Bascom) by **4pm** sharp. Assignments may be submitted later than this by email to Alyssa, but no later than 4pm on Friday; in this case, the grade will be reduced by 5%.
Assignments submitted later than Friday at 4pm will not be graded.

Please print and attach this page as the first page of your submitted problem set.

PROBLEM	**GRADE**
5.1	
5.2	
5.3	
5.4	
Total	

Please read the following statement and sign below:

I understand that I am not allowed to use the internet to assist with this assignment. I also understand that I must write down the final version of my assignment in isolation from any other person. I pledge to abide by the Williams honor code.

SIGNATURE:______________________________

Problem Set 5

5.1 Irreducible fun!

(a) Is $x^6 + x + 1$ irreducible over $\mathbb{Q}$? Prove or disprove.

(b) Is $x^4 + 4$ irreducible over $\mathbb{Q}$? Prove or disprove.

(c) Give five different proofs that $x^3 - 2$ is irreducible over $\mathbb{Q}$. [*Please don't use Eisenstein more than once.*]

(d) Can you produce an example of a polynomial f and a prime p such that $[f]$ is irreducible over $\mathbb{F}_p$, but f is reducible over $\mathbb{Q}$?

(e) Prove that $1 + t + t^2 + \cdots + t^{n-1}$ is irreducible iff n is prime.

5.2 True facts about field extensions.

(a) Suppose K and L are fields, and that there exists a ring homomorphism $\varphi : K \to L$. Prove that L is a field extension of K.

(b) Prove that $[L : K] = 1$ if and only if $L \simeq K$.

(c) Suppose L/K is a field extension with char $K \neq 2$. Prove that $[L : K] = 2$ if and only if $\exists \alpha \in L \setminus K$ such that $L = K(\alpha)$ and $\alpha^2 \in K$.

(d) Suppose L/K is a field extension with the property that *every* $\alpha \in L$ is algebraic over K. Prove that any ring R lying between K and L (i.e. $K \subseteq R \subseteq L$) must be a field.

(e) Suppose $\alpha \in L/K$ is transcendental over K. Prove that $K(\alpha) \not\simeq K[\alpha]$.

5.3 Minimal polynomials, maximal awesome. [*Throughout, we use the following notation: if α is algebraic over K, we denote the minimal polynomial of α over K by m_α. By convention, $m_\alpha \in K[t]$ is monic.*]

(a) Suppose $\alpha \in \mathbb{C}$ is the root of some *monic* polynomial $f \in \mathbb{Z}[t]$. Prove that $m_\alpha \in \mathbb{Z}[t]$.

(b) Find the minimal polynomial for $\sqrt{2} + \sqrt{-5}$ over $\mathbb{Q}$.

(c) Find the minimal polynomial for $\sqrt{2} + \sqrt{-5}$ over $\mathbb{R}$.

(d) Find the minimal polynomial for $e^{2\pi i/5}$ over $\mathbb{Q}$.

(e) Given α algebraic over K, suppose m_α has odd degree. Prove that $K(\alpha^2) = K(\alpha)$.

5.4 The goal of this exercise is to show that polynomials can't have too many roots in a field. More precisely:

Theorem 1. *Let K be a field, and suppose $f \in K[x]$ is nonconstant. Consider the collection of all its roots in K:*

$$\mathcal{Z}_f := \{\alpha \in K : f(\alpha) = 0\}.$$

Then $|\mathcal{Z}_f| \leq \deg f$.

(a) For any $f \in K[x]$ and any $\alpha \in K$, prove that $(x - \alpha) \mid \big(f(x) - f(\alpha)\big)$. (In other words, prove that $\frac{f(x)-f(\alpha)}{x-\alpha} \in K[x]$.)

(b) Suppose $\alpha \in \mathcal{Z}_f$. Prove that $f(x) = (x - \alpha)g(x)$ for some $g \in K[x]$, and that $\mathcal{Z}_f \subseteq \mathcal{Z}_g \cup \{\alpha\}$.

(c) Prove the theorem.

(d) Does the theorem hold if K is a ring, rather than a field? Justify your answer.