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MATH 394 : GALOIS THEORY

Problem Set 7 – due Monday, April 22nd

INSTRUCTIONS:

You should aim to submit this assignment to me in person at the start of Monday's class. Late assignments can be left in the mailbox outside my office until 10pm on Monday (incurring a small penalty, as described in the course syllabus). Assignments will not be accepted after 10pm on Monday under any circumstances.

7.1 Suppose V is a finite-dimensional vector space over a field K of characteristic 0. Show that for any finite collection of proper subspaces $W_1, W_2, \ldots, W_n$ of V, we have $V \neq \bigcup_{1 \leq i \leq n} W_i$. [*Hint: There exists* $x \in V \setminus W_1$ *and* $y \in V \setminus W_2$. *Start by proving that some linear combination of* x *and* y *doesn't belong to* $W_1 \cup W_2$.]

7.2 In class we proved that if L/K is a finite Galois extension, then there exists some α such that $L = K(\alpha)$. In this case, α is called a *primitive element* of the extension and L/K is said to be a *simple* extension.

(a) Why is $\mathbb{Q}(\omega, \sqrt[3]{2})/\mathbb{Q}$ Galois? Determine (with proof!) a primitive element of this extension.

(b) Prove that $\mathbb{Q}(i, \sqrt{7})/\mathbb{Q}$ is a simple extension.

7.3 Finiteness of the automorphism group.

(a) If L/K is a finite extension, prove that $\mathrm{Aut}(L/K)$ is a finite group.

(b) Given $f \in K[t]$ of degree n, let L/K be a splitting field of f. Prove that $|\mathrm{Aut}(L/K)| \leq n!$.

7.4 Given a separable polynomial $f \in K[t]$. The *Galois group of* f is defined to be the Galois group of the extension L/K, where L is the splitting field of f.

(a) Prove that any element of the Galois group of f is a permutation of the roots of f. Deduce that the Galois group of f embeds into S_n, where $n := \deg f$.

(b) Determine the Galois group of $x^4 - 5x^2 + 6$ over $\mathbb{Q}$.

7.5 Exploring Galois extensions.

(a) Suppose $[L : K] = 2$, where char $K \neq 2$. Must L/K be a Galois extension? Prove or disprove.

(b) Find an example of a tower of finite extensions $L/F/K$ such that both L/F and F/K are Galois, but L/K is not. Prove all your assertions. [*Hint: Start with* $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$.]

7.6 The goal of this exercise is to determine $\mathrm{Aut}(\mathbb{R}/\mathbb{Q})$.

(a) Prove that every $\sigma \in \mathrm{Aut}(\mathbb{R}/\mathbb{Q})$ is order-preserving, i.e. that $\sigma(a) < \sigma(b)$ whenever $a < b$.

(b) Prove that any $\sigma \in \mathrm{Aut}(\mathbb{R}/\mathbb{Q})$ is continuous on $\mathbb{R}$.

(c) Determine $\mathrm{Aut}(\mathbb{R}/\mathbb{Q})$.