Instructor: Leo Goldmakher

NAME: ______________________________

Williams College
Department of Mathematics and Statistics

MATH 394 : GALOIS THEORY

Problem Set 8 – due Thursday, April 19th

INSTRUCTIONS:
This assignment must be turned in to my mailbox (on the right as you enter Bascom) by **4pm** sharp. Assignments may be submitted later than this by email to Alyssa, but no later than 4pm on Friday; in this case, the grade will be reduced by 5%.
Assignments submitted later than Friday at 4pm will not be graded.

Please print and attach this page as the first page of your submitted problem set.

PROBLEM	GRADE
8.1	
8.2	
8.3	
8.4	
8.5	
8.6	
Total	

Please read the following statement and sign below:

I understand that I am not allowed to use the internet to assist with this assignment. I also understand that I must write down the final version of my assignment in isolation from any other person. I pledge to abide by the Williams honor code.

SIGNATURE:______________________________

Problem Set 8

8.1 Suppose V is a finite-dimensional vector space over a field K of characteristic 0. Show that for any finite collection of proper subspaces $W_1, W_2, \ldots, W_n$ of V, we have $V \neq \bigcup_{1 \le i \le n} W_i$. [*Hint: There exists $x \in V \setminus W_1$ and $y \in V \setminus W_2$. Start by proving that some linear combination of x and y doesn't belong to $W_1 \cup W_2$.*]

8.2 In class we proved that if L/K is a finite Galois extension, then there exists some α such that $L = K(\alpha)$. In this case, α is called a *primitive element* of the extension and L/K is said to be a *simple* extension.

(a) Why is $\mathbb{Q}(\omega, \sqrt[3]{2})/\mathbb{Q}$ Galois? Find a primitive element of this extension.

(b) Prove that $\mathbb{Q}(i, \sqrt{7})/\mathbb{Q}$ is a simple extension.

8.3 Finiteness of the automorphism group.

(a) If L/K is a finite extension, prove that $\mathrm{Aut}(L/K)$ is a finite group.

(b) Given $f \in K[t]$ of degree n, let L/K be a splitting field of f. Prove that $|\mathrm{Aut}(L/K)| \le n!$.

8.4 Given a separable polynomial $f \in K[t]$. The *Galois group of f* is defined to be the Galois group of the extension L/K, where L is the splitting field of f.

(a) Prove that any element of the Galois group of f is a permutation of the roots of f. Deduce that the Galois group of f embeds into S_n, where $n := \deg f$.

(b) Determine the Galois group of $x^4 - 5x^2 + 6$ over $\mathbb{Q}$.

8.5 Exploring Galois extensions.

(a) Suppose $[L : K] = 2$, where $\mathrm{char}\, K \neq 2$. Must L/K be a Galois extension? Prove or disprove.

(b) Find an example of a tower of finite extensions $L/F/K$ such that both L/F and F/K are Galois, but L/K is not. Prove all your assertions. [*Hint: Start with $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$.*]

8.6 The goal of this exercise is to determine $\mathrm{Aut}(\mathbb{R}/\mathbb{Q})$.

(a) Prove that every $\sigma \in \mathrm{Aut}(\mathbb{R}/\mathbb{Q})$ is order-preserving, i.e. that $\sigma(a) < \sigma(b)$ whenever $a < b$.

(b) Prove that any $\sigma \in \mathrm{Aut}(\mathbb{R}/\mathbb{Q})$ is continuous on $\mathbb{R}$.

(c) Determine $\mathrm{Aut}(\mathbb{R}/\mathbb{Q})$.