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Williams College
Department of Mathematics and Statistics

MATH 394 : GALOIS THEORY

Problem Set 9 – due Thursday, April 26th

INSTRUCTIONS:
This assignment must be turned in to my mailbox (on the right as you enter Bascom) by **4pm** sharp. Assignments may be submitted later than this by email to Alyssa, but no later than 4pm on Friday; in this case, the grade will be reduced by 5%.
Assignments submitted later than Friday at 4pm will not be graded.

Please print and attach this page as the first page of your submitted problem set.

PROBLEM	**GRADE**
9.1	
9.2	
9.3	
9.4	
9.5	
9.6	
Total	

Please read the following statement and sign below:

I understand that I am not allowed to use the internet to assist with this assignment. I also understand that I must write down the final version of my assignment in isolation from any other person. I pledge to abide by the Williams honor code.

SIGNATURE:______________________________

Problem Set 9

9.1 Let $f(x) = x^4 - 3x^2 + 4$, and let K denote the splitting field of f over $\mathbb{Q}$. Prove (verify) all parts of the fundamental theorem of Galois theory for the extension $K/\mathbb{Q}$.

9.2 Let ζ_5 denote the 5th root of unity, i.e. $\zeta_5 := e^{2\pi i/5}$.

(a) Prove that $\mathbb{Q}(\zeta_5)/\mathbb{Q}$ is a Galois extension.

(b) Determine the Galois group of the extension $\mathbb{Q}(\zeta_5)/\mathbb{Q}$. What familiar group is it isomorphic to?

9.3 Suppose α is algebraic over K, and let $m_\alpha \in K[x]$ denote the minimal polynomial of α over K. If β is a root of m_α, must m_α be the minimal polynomial of β over K? Prove or disprove.

9.4 In class, we discussed enlarging a given extension to be Galois. (Moreover, we used this in an essential way in our proof of the Fundamental Theorem of Algebra.) The goal of this problem is to put this on a rigorous footing. We will need

Definition. A field extension L/K is *separable* iff every element $\alpha \in L/K$ has separable minimal polynomial over K. (In this case, we also refer to the element α as being separable.)

(a) Prove that any finite separable extension F/K can be enlarged to a finite Galois extension of K (i.e., prove there exists a finite extension L/F such that L/K is Galois).

(b) Prove that any finite extension of a characteristic 0 field can be enlarged to a finite Galois extension.

9.5 Prove the following result we asserted in class:

Lemma. $f, g \in K[x]$ are coprime if and only if f, g have no common roots in a splitting field L/K of f.

[*Recall that f and g are* coprime *iff the only common factors of f and g in $K[x]$ are constants.*]

9.6 Our goal is to complete the proof of the equivalence of the three characterizations of Galois-ity, by proving B $\implies$ A. More precisely, we need to prove that if L/K is the splitting field of some separable polynomial $f \in K[x]$, then $|\mathrm{Aut}(L/K)| = [L : K]$. For ease of notation, let A denote the set of all roots of f and $G := \mathrm{Aut}(L/K)$.

(a) Prove that $\sigma(A) = A$ for all $\sigma \in G$. [*Note: this doesn't mean σ fixes every element of A!*]

(b) Given $\alpha \in A$, an intermediate field F lying between K and L, and a K-homomorphism $\sigma : F \to L$. Prove there are precisely $[F(\alpha) : F]$ ways to lift σ to a K-homomorphism $\widehat{\sigma} : F(\alpha) \to L$. [*By 'K-homomorphism' I mean a homomorphism which fixes all elements of K. By 'lift' I mean define a $\widehat{\sigma}$ which agrees with σ on all inputs from F.*]

(c) Prove that $|G| = [L : K]$. [*Hint: Note that $L = K(A)$ (why?). Now use (b).*]