Instructor: Leo Goldmakher

NAME: ______________________________

Williams College
Department of Mathematics and Statistics

MATH 394 : GALOIS THEORY

Problem Set 11 – due Thursday, May 11th

INSTRUCTIONS:
This assignment must be turned in to my mailbox (on the right as you enter Bascom) by **4pm** sharp. Assignments may be submitted later than this by email to Alyssa, but no later than 4pm on Friday; in this case, the grade will be reduced by 5%.
Assignments submitted later than Friday at 4pm will not be graded.

Please print and attach this page as the first page of your submitted problem set.

PROBLEM	GRADE
11.1	
11.2	
11.3	
11.4	
11.5	
11.6	
Total	

Please read the following statement and sign below:

I understand that I am not allowed to use the internet to assist with this assignment. I also understand that I must write down the final version of my assignment in isolation from any other person. I pledge to abide by the Williams honor code.

SIGNATURE:______________________________

Problem Set 11

11.1 Given a group G, recall that the *conjugacy class of* $a \in G$ is defined to be

$$C(a) := \{gag^{-1} : g \in G\}.$$

(a) Prove that $N \trianglelefteq G$ if and only if $\exists A$ with $\{e\} \subseteq A \subseteq G$ such that $N = \bigcup_{a \in A} C(a)$.

(b) It turns out (doing a brute force calculation) that the alternating group A_5 can be partitioned into five distinct conjugacy classes:

- $C(e)$, which consists of one element;
- $C\big((1\ 2\ 3\ 4\ 5)\big)$, which consists of 12 elements;
- $C\big((2\ 1\ 3\ 4\ 5)\big)$, which consists of 12 elements;
- $C\big((1\ 2)(3\ 4)\big)$, which consists of 15 elements; and
- $C\big((1\ 2\ 3)\big)$, which consists of 20 elements.

Use this information to prove that A_5 is simple. [*Your proof should be very short.*]

11.2 In class I drew an analogy between unique factorization in $\mathbb{Z}$ and the Jordan-Hölder theorem. The goal of this problem is to explore this further.

(a) Determine (with proof) two non-isomorphic groups G such that

- G has a normal subgroup N which is the cyclic group of order 6, and
- G/N is the cyclic group of order 2.

(b) Determine (with proof) two non-isomorphic groups which have the same list of composition factors.

(c) Prove that Jordan-Hölder generalizes the fundamental theorem of arithmetic. In other words, prove that Jordan-Hölder implies that any $n \in \mathbb{Z}$ has a unique factorization into primes.

11.3 Let ζ_n denote the principal n^{th} root of unity, i.e. $\zeta_n = e^{2\pi i/n}$.

(a) Prove that $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is Galois.

(b) Let $G := \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$. Prove that G embeds into $(\mathbb{Z}/n\mathbb{Z})^\times$, the group of units of $\mathbb{Z}/n\mathbb{Z}$. Deduce that G is abelian. [*Recall that a* unit *of a ring is an element which is invertible under multiplication. By* embed *I mean there exists an injective homomorphism.*]

11.4 The goal of this problem is to determine $\mathrm{Gal}(f)$ for $f(x) = x^5 - 4x - 2$. [Review problem **8.4(a)**.]

(a) Prove that $\mathrm{Gal}(f)$ contains an element of order 5. [*Hint: use Cauchy's theorem, aka problem **3.5**.*]

(b) Prove that $\mathrm{Gal}(f)$ contains an element of order 2. [*Hint: f has two roots in $\mathbb{C}$ and three in $\mathbb{R}$.*]

(c) Prove that $\mathrm{Gal}(f) \simeq S_5$.

11.5 Prove that the normal series $G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n := \{e\}$ is abelian (i.e. that all the quotients G_i/G_{i+1} are abelian) if and only if $[G_i, G_i] \trianglelefteq G_{i+1}$ for all i. [*Hint: see problem **2.1**.*]

11.6 Given an irreducible $f \in \mathbb{Q}[x]$ which has a root that can be expressed in terms of radicals. Prove that every root of f can be expressed in terms of radicals.