UNIVERSITY OF TORONTO SCARBOROUGH

MATA31H3 : Calculus for Mathematical Sciences I

FINAL EXAMINATION

December 20, 2012

Duration – 3 hours
Aids: none

NAME (PRINT): ______________________________
Last/Surname First/Given Name (and nickname)

STUDENT NO: KEY

TUTORIAL: ______________________________
Tutorial section # Name of TA

Qn. #	Value	Score
COVER PAGE	5	
1	25	
2	10	
3	15	
4	10	
5	10	
6	25	
Total	100	

TOTAL: ____________

Please read the following statement and sign below:

I understand that any breach of academic integrity is a violation of The Code of Behaviour on Academic Matters. By signing below, I pledge to abide by the Code.

SIGNATURE:______________________________

(1) (a) (15 points) Prove that $\lim_{x\to 2} x^2 - 3x + 3 = 1$. For this part of the problem you may *not* use any theorems from lecture.

Fix $\epsilon > 0$, and suppose x satisfies

(*) $$0 < |x-2| < \min\left\{1, \frac{\epsilon}{2}\right\}.$$

It follows that $|x-2| < 1$, i.e. that $-1 < x-2 < 1$. Thus, for all x satisfying (*), we have $0 < x-1 < 2$. On the other hand, (*) implies that $|x-2| < \frac{\epsilon}{2}$, so

$$|(x^2 - 3x + 3) - 1| = |x-1| \cdot |x-2| < 2 \cdot \frac{\epsilon}{2} = \epsilon.$$

This proves the claim. □

(b) (10 points) Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies $x - 1 \le f(x) \le x^2 - 3x + 3$ for all $x \in \mathbb{R}$. Determine (with proof) the value of $\lim_{x\to 2} f(x)$. For this part of the problem you *may* use any theorems from lecture.

First, note that $\lim_{x\to 2} x - 1 = 1$. [For any $\epsilon > 0$, we have $0 < |x-2| < \epsilon \implies |(x-1)-1| = |x-2| < \epsilon$.] Next, from part (a) we know that $\lim_{x\to 2} x^2 - 3x + 3 = 1$. The squeeze theorem therefore implies that $\lim_{x\to 2} f(x)$ exists and is equal to 1.

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(2) Consider the function

$$\begin{aligned} g: \mathbb{R} &\longrightarrow [0,1) \\ x &\longmapsto \frac{x^2}{1+x^2} \end{aligned}$$

(a) (5 points) Is g injective? If so, prove it; if not, provide a counterexample.

No, g is not injective. For example, $g(1) = \frac{1}{2} = g(-1)$.

(b) (5 points) Is g surjective? If so, prove it; if not, provide a counterexample.

Yes, g is surjective. In other words, for every $y \in [0,1)$, there exists some real number which gets mapped to y by the function g. To see this, pick an arbitrary $y \in [0,1)$. Then $\sqrt{\frac{y}{1-y}} \in \mathbb{R}$, since $y \geq 0$ and $1-y > 0$. Moreover,

$$g\left(\sqrt{\frac{y}{1-y}}\right) = \frac{\frac{y}{1-y}}{1+\frac{y}{1-y}} = \frac{\frac{y}{1-y}}{\frac{1}{1-y}} = y.$$

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(3) (15 points) Let

$$h(x) := \begin{cases} \sin x & \text{if } x \in \mathbb{Q} \\ 3 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Prove that $\lim_{x \to 2} h(x)$ does not exist.

We proceed by contradiction. Suppose that the limit does exist, say,

$$\lim_{x \to 2} h(x) = L.$$

It follows that there exists some constant $\delta > 0$ such that

(†) $$|h(x) - L| < \frac{1}{2} \qquad \forall x \in (2, 2+\delta).$$

From the density theorem, we know that there exists a rational number $a \in (2, 2+\delta)$ and an irrational number $b \in (2, 2+\delta)$. The inequality (†) thus implies that

$$|h(a) - L| < \frac{1}{2} \qquad \text{and} \qquad |h(b) - L| < \frac{1}{2}$$

so that triangle inequality implies

(‡) $$|\sin a - 3| = |(\sin a - L) + (L - 3)| \leq |\sin a - L| + |L - 3| < 1.$$

On the other hand, since $\sin a \leq 1$, we have

$$|\sin a - 3| = 3 - \sin a \geq 2,$$

which contradicts (‡). □

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(4) (10 points) Prove that

$$\lim_{x\to\infty} \frac{3x+5}{x-2} = 3.$$

Pick $\epsilon > 0$. Then for all $x \geq \frac{100}{\epsilon} + 2$ we have

$$\begin{aligned}\left|\frac{3x+5}{x-2} - 3\right| &= \left|\frac{11}{x-2}\right| \\ &= \frac{11}{x-2} \qquad \text{(since } x > 2\text{)} \\ &\leq \frac{11}{100/\epsilon} \\ &= \frac{11}{100}\,\epsilon \\ &< \epsilon.\end{aligned}$$

□

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(5) (10 points) Suppose $F : \mathbb{R} \longrightarrow \mathbb{R}$ (i.e. $F(x)$ is defined for all $x \in \mathbb{R}$), and that $\lim_{x \to 1} F(x) = 2$. Prove that $F(x)$ is bounded in some (nonempty) neighbourhood of 1. [For this problem, you must prove directly from definitions; i.e. you may *not* refer to any theorems.]

By the definition of the limit, there exists some number $\delta > 0$ such that

$$|F(x) - 2| < 1 \qquad \forall x \in (1 - \delta, 1 + \delta) \backslash \{1\}.$$

In particular, it follows that

$$|F(x)| < 3 \qquad \forall x \in (1 - \delta, 1 + \delta) \backslash \{1\}.$$

Let $M := \max\{3, |F(1)|\}$ (note that $F(1) \in \mathbb{R}$ by hypothesis). Then we see that in the open interval $(1 - \delta, 1 + \delta)$, we have

$$|F(x)| \leq M.$$

This shows that $F(x)$ is bounded in a nonempty neighbourhood of 1. $\square$

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(6) (a) (10 points) Use induction to prove that for all $N \in \mathbb{N}$,

$$\sum_{n=1}^{N} \frac{1}{2^n} = 1 - \frac{1}{2^N}.$$

The claim is easily verified for $N = 1$. Suppose it is true for $N = k$, i.e.

$$\sum_{n=1}^{k} \frac{1}{2^n} = 1 - \frac{1}{2^k}.$$

Adding $1/2^{k+1}$ to both sides, we deduce that

$$\sum_{n=1}^{k+1} \frac{1}{2^n} = 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^{k+1}}.$$

In other words, whenever the claim holds with $N = k$, then it continues to hold with $N = k + 1$. By induction, we conclude that the identity holds for all natural numbers N. □

(b) (15 points) Recall that $\mathbb{R}_{\geq 0} := \{r \in \mathbb{R} : r \geq 0\}$. Given a function $G : \mathbb{N} \longrightarrow \mathbb{R}_{\geq 0}$, define

$$\sum_{n=1}^{\infty} G(n) := \sup \left\{ \sum_{n=1}^{N} G(n) : N \in \mathbb{N} \right\},$$

if this supremum exists. Determine (with proof) the value of

$$\sum_{n=1}^{\infty} \frac{1}{2^n}.$$

Claim. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$

Proof. Combining part (a) with the definition given in the problem, we have

$$\sum_{n=1}^{\infty} \frac{1}{2^n} := \sup \left\{ 1 - \frac{1}{2^N} : N \in \mathbb{N} \right\}$$

if this supremum exists. It therefore suffices to prove that

$$\sup \left\{ 1 - \frac{1}{2^N} : N \in \mathbb{N} \right\} = 1.$$

It is clear that 1 is an upper bound of the set $S := \left\{ 1 - \frac{1}{2^N} : N \in \mathbb{N} \right\}$; moreover, S is clearly nonempty. The Completeness Property of $\mathbb{R}$ therefore implies that the supremum of S exists, so our only remaining task is to prove that 1 is the *least* upper bound of S. To do this, we will show that anything smaller than 1 cannot be an upper bound of S. Indeed, pick any $\alpha < 1$. Then $1 - \alpha > 0$, whence $\frac{1}{1-\alpha} \in \mathbb{R}$. The Archimedean Property guarantees the existence of a natural number $M > \frac{1}{1-\alpha}$. It follows that

$$2^M > M > \frac{1}{1 - \alpha},$$

whence we deduce that $\alpha < 1 - \frac{1}{2^M}$. This immediately implies that α is not an upper bound of S. We therefore conclude that 1 is the least upper bound of S, as claimed. □

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Total Marks = 100 points