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UNIVERSITY OF TORONTO SCARBOROUGH

MATA31H3 F : Calculus for Mathematical Sciences I

REFERENCE SHEET

Properties of real numbers

Throughout this course, we assume that there exists a set $\mathbb{R}$ endowed with two binary operations $+$ and $\cdot$ satisfying all the following properties:

Algebraic Properties:

(A1) $\forall a, b \in \mathbb{R}$, we have $a + b = b + a$.

(A2) $\forall a, b, c \in \mathbb{R}$, we have $a + (b + c) = (a + b) + c$.

(A3) There exists a number $0 \in \mathbb{R}$ such that $\forall a \in \mathbb{R}$, $a + 0 = a$.

(A4) For each $a \in \mathbb{R}$ there exists a number $-a \in \mathbb{R}$ such that $a + (-a) = 0$.

(M1) $\forall a, b \in \mathbb{R}$, we have $a \cdot b = b \cdot a$.

(M2) $\forall a, b, c \in \mathbb{R}$, we have $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

(M3) There exists a number $1 \in \mathbb{R}$ such that $\forall a \in \mathbb{R}$, $a \cdot 1 = a$. Moreover, $1 \neq 0$.

(M4) For each $a \in \mathbb{R}\backslash\{0\}$, there exists a number $a^{-1} \in \mathbb{R}$ such that $a \cdot a^{-1} = 1$.

(D) $\forall a, b, c \in \mathbb{R}$, we have $a \cdot (b + c) = a \cdot b + a \cdot c$.

Order Properties: There exists a special subset $\mathcal{P} \subseteq \mathbb{R}$ such that

(O1) $0 \notin \mathcal{P}$

(O2) For all $a \in \mathbb{R}\backslash\{0\}$, $a \notin \mathcal{P}$ iff $-a \in \mathcal{P}$.

(O3) If $a, b \in \mathcal{P}$, then $a + b \in \mathcal{P}$ and $a \cdot b \in \mathcal{P}$.

Completeness Property

(C) If $\mathcal{A} \subseteq \mathbb{R}$ is nonempty and bounded above, then there exists a *least* upper bound of $\mathcal{A}$.

NOTATION

$\sup \mathcal{A}$, the *supremum* of $\mathcal{A}$, is the least upper bound of $\mathcal{A}$ (if it exists).

$\inf \mathcal{A}$, the *infimum* of $\mathcal{A}$, is the greatest lower bound of $\mathcal{A}$ (if it exists).

The symbol $a - b$ means $a + (-b)$. The symbol $\frac{a}{b}$ means $a \cdot b^{-1}$.

For all $n \in \mathbb{N}$, we define the symbol $a^{n+1} := a^n \cdot a$, where $a^1 := a$.

Given $a, b \in \mathbb{R}$, the notation $a > b$ means that $a + (-b) \in \mathcal{P}$, and $b < a$ is another way of writing $a > b$. The symbol $a \geq b$ means that $a + (-b) \in \mathcal{P} \cup \{0\}$, and $b \leq a$ means the same as $a \geq b$.

UNIVERSITY OF TORONTO SCARBOROUGH

MATA31H3 : Calculus for Mathematical Sciences I

MIDTERM EXAMINATION # 2

November 9, 2012

Duration – 2 hours
Aids: none

NAME (PRINT): ______________________________
Last/Surname   First/Given Name (and nickname)

STUDENT NO: KEY

TUTORIAL: ______________________________
Tutorial section #   Name of TA

Qn. #	Value	Score
COVER PAGE	5	
1	20	
2	20	
3	12	
4	18	
5	15	
6	10	
Total	100	

TOTAL: ____________

Please read the following statement and sign below:

I understand that any breach of academic integrity is a violation of The Code of Behaviour on Academic Matters. By signing below, I pledge to abide by the Code.

SIGNATURE:______________________________

(1) Prove each of the following, explicitly justifying every step of your proof with the appropriate property of $\mathbb{R}$.

(a) (8 points) $2 \cdot 2 = 4$.

We have

$$\begin{aligned} 2 \cdot 2 &= 2 \cdot (1+1) && \text{by definition of 2} \\ &= 2 \cdot 1 + 2 \cdot 1 && \text{by (D)} \\ &= 2 + 2 && \text{by (M3)} \\ &= (1+1) + (1+1) && \\ &= 1+1+1+1 && \text{by (A2)} \\ &= 4 && \text{by definition of 4.} \end{aligned}$$

(b) (12 points) If $a \in (0, 2)$, then $a^2 < 4$.

If $a \in (0, 2)$, then $a > 0$ and $a < 2$, or in other words,

$$a \in \mathcal{P} \qquad \text{and} \qquad 2 - a \in \mathcal{P}.$$

By (O3), we deduce that $a + a \in \mathcal{P}$. Note that

$$\begin{aligned} (2-a) + (a+a) &= 2 + (-a) + a + a && \text{by (A2)} \\ &= 2 + 0 + a && \text{by (A4)} \\ &= 2 + a && \text{by (A3).} \end{aligned}$$

Since $2 - a \in \mathcal{P}$ and $a + a \in \mathcal{P}$, (O3) implies that $2 + a \in \mathcal{P}$. Applying (O3) yet again to the two numbers $2 - a$ and $2 + a$, we deduce that

(*) $$(2-a) \cdot (2+a) \in \mathcal{P}.$$

Finally, we have

$$\begin{aligned} (2-a)\cdot(2+a) &= \Big((2-a)\cdot 2\Big) + \Big((2-a)\cdot a\Big) && \text{by (D)} \\ &= \Big(2\cdot(2-a)\Big) + \Big(a\cdot(2-a)\Big) && \text{by (M1)} \\ &= \Big(2\cdot 2 + 2\cdot(-a)\Big) + \Big(a\cdot 2 + a\cdot(-a)\Big) && \text{by (D)} \\ &= 2\cdot 2 + 2\cdot(-a) + a\cdot 2 + a\cdot(-a) && \text{by (A2)} \\ &= 4 + 2\cdot(-a) + a\cdot 2 + a\cdot(-a) && \text{by part (a)} \\ &= 4 + 2\cdot(-a) + 2\cdot a + a\cdot(-a) && \text{by (M1)} \\ &= 4 + 2\cdot\Big((-a) + a\Big) + a\cdot(-a) && \text{by (D)} \\ &= 4 + 2\cdot\Big(a + (-a)\Big) + a\cdot(-a) && \text{by (A1)} \\ &= 4 + 2\cdot 0 + a\cdot(-a) && \text{by (A4)} \\ &= 4 + 0 + a\cdot(-a) && \text{by proof in lecture} \\ &= 4 + \Big(a^2 + (-a^2)\Big) + a\cdot(-a) && \text{by (A4)} \\ &= 4 + \Big((-a^2) + a^2\Big) + a\cdot(-a) && \text{by (A1)} \\ &= 4 - a^2 + a^2 + a\cdot(-a) && \text{by (A2)} \\ &= 4 - a^2 + a\cdot\Big(a + (-a)\Big) && \text{by (D)} \\ &= 4 - a^2 + a\cdot 0 && \text{by (A4)} \\ &= 4 - a^2 + 0 && \text{by proof from lecture} \\ &= 4 - a^2 && \text{by (A3).} \end{aligned}$$

Combining this with (*) above, we see that $4 - a^2 \in \mathcal{P}$, or equivalently, $a^2 < 4$. $\square$

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(2) (20 points) Prove that $\sqrt{5}$ exists. In other words, prove that there exists a positive number $x \in \mathbb{R}$ satisfying $x^2 = 5$. [You do *not* need to explicitly use the algebraic and order properties in this problem, or in any of the following problems.]

Let $\mathcal{A} := \{x > 0 : x^2 < 5\}$. Clearly $\mathcal{A} \neq \emptyset$ (since $2 \in \mathcal{A}$). Also, observe that $\mathcal{A}$ is bounded above by 3, since

$$3 < x \implies x^2 > 9 \implies x \notin \mathcal{A}.$$

Thus, by Completeness, $\sup \mathcal{A}$ exists. Let

$$\alpha = \sup \mathcal{A}.$$

I claim that $\alpha^2 = 5$; this will prove the claim.

Suppose $\alpha^2 < 5$. Note that in this case, $\alpha \in \mathcal{A}$. (α is clearly positive, since it is an upper bound of $\mathcal{A}$ and $2 \in \mathcal{A}$.) Since $\alpha^2 < 5$, we have $\frac{2\alpha+1}{5-\alpha^2} \in \mathbb{R}$. The Archimedean Property therefore implies the existence of a natural number n such that

$$n > \frac{2\alpha + 1}{5 - \alpha^2},$$

from which we deduce that

$$\frac{2\alpha}{n} + \frac{1}{n} < 5 - \alpha^2.$$

Since $\frac{2\alpha}{n} + \frac{1}{n^2} \leq \frac{2\alpha}{n} + \frac{1}{n}$ for all $n \in \mathbb{N}$, we see that

$$\frac{2\alpha}{n} + \frac{1}{n^2} < 5 - \alpha^2,$$

or in other words, that

$$(\alpha + 1/n)^2 < 5.$$

But this means that $\alpha + 1/n \in \mathcal{A}$, contradicting that α is an upper bound of $\mathcal{A}$.

Next, suppose instead that $\alpha^2 > 5$. Then $\frac{2\alpha}{\alpha^2-5} \in \mathbb{R}$, so by Archimedean Property,

$$\exists n \in \mathbb{N} \text{ such that } n > \frac{2\alpha}{\alpha^2 - 5}.$$

It follows that

$$\alpha^2 - \frac{2\alpha}{n} > 5,$$

whence

$$(\alpha - 1/n)^2 > 5.$$

This implies that $\alpha - 1/n$ is an upper bound of $\mathcal{A}$ (for the same reason that 3 was an upper bound). But this contradicts that α is the *least* upper bound of $\mathcal{A}$.

Thus, both $\alpha^2 < 5$ and $\alpha^2 > 5$ are impossible. By trichotomy, we must have $\alpha^2 = 5$. Since α is positive, this concludes the proof. $\square$

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(3) (12 points) Prove that 1 is the only real number which satisfies

$$|x-1| < \frac{1}{n^2}$$

for every $n \in \mathbb{N}$.

From the definition of absolute value, we know that $|x-1| \geq 0$ for all $x \in \mathbb{R}$. Suppose $|x-1| \neq 0$. Then $\frac{1}{|x-1|} \in \mathbb{R}$, so the Archimedean Property implies the existence of an $m \in \mathbb{N}$ such that

$$m > \frac{1}{|x-1|}.$$

Since $m^2 \geq m$ for all $m \in \mathbb{N}$, we deduce that

$$m^2 > \frac{1}{|x-1|},$$

which implies that

$$|x-1| > \frac{1}{m^2}.$$

We have therefore shown that if $|x-1| \neq 0$, then x cannot satisfy the given inequality for every $n \in \mathbb{N}$. Thus, if x does satisfy the inequality for every n, we must have $|x-1| = 0$, i.e. $x = 1$. □

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(4) (18 points) Given $x, y \in \mathbb{R}$ such that $0 < x < y$. Prove that there exist $a, b \in \mathbb{N}$ such that a is even, b is odd, and $x < \frac{a}{b} < y$.

Since $y > x$, we see that $\frac{2}{y-x} \in \mathbb{R}$. By the Archimedean Property, we know that

$$\exists n \in \mathbb{N} \text{ such that } n > \frac{2}{y-x}.$$

Moreover, we may assume that n is odd; if not, add 1 to it (it will still be bigger than $\frac{2}{y-x}$). Note that

(**) $$ny > 2 + nx.$$

Let

$$\mathcal{A} := \{k \in \mathbb{N} : k \text{ is even and } k \geq ny\}.$$

By the Well-Ordering Property, $\mathcal{A}$ has a smallest element m. I claim that $m - 2$ is an even natural number which is between nx and ny. The first part of this claim is easy: $m \in \mathcal{A}$ means m is an even integer, so $m - 2$ must also be an even integer. Moreover, (**) implies that $m > 2$, so $m - 2$ must be a natural number. It thus remains to show that $m - 2 \in (nx, ny)$.

Since m is the least element of $\mathcal{A}$, $m - 2 \notin \mathcal{A}$. As discussed above, $m - 2$ is an even natural number, so the only way it could not belong to $\mathcal{A}$ is if

$$m - 2 < ny.$$

On the other hand, since $m \in \mathcal{A}$, $m \geq ny$, so (**) implies

$$m - 2 \geq ny - 2 > nx.$$

Combining the above inequalities, we conclude that

$$nx < m - 2 < ny,$$

which implies

$$x < \frac{m-2}{n} < y.$$

Since m is even, $m - 2$ must also be even, and n is odd from the outset. This concludes the proof. □

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(5) (15 points) Suppose A and B are nonempty subsets of $\mathbb{R}$ which satisfy

$$a \le b \quad \text{for all } a \in A \text{ and } b \in B.$$

Prove that $\sup A \le \inf B$. [*Hint: you may find it helpful to prove that* $\sup A \le b$ *for any* $b \in B$.]

This problem is an example from the reading.

Pick any $b \in B$. By hypothesis, $a \le b$ for all $a \in A$, which means that b is an upper bound of A. Also, $A \ne \emptyset$. Completeness therefore implies that $\sup A$ exists. Moreover, since b is an upper bound of A, we see that $\sup A \le b$.

In the above argument, the choice of $b \in B$ is arbitrary. In other words, we've proved that for *every* $b \in B$,

$$\sup A \le b.$$

Thus, $\sup A$ is a lower bound on B. Since B is nonempty, Completeness implies the existence of $\inf B$. Since $\sup A$ is a lower bound of B, and $\inf B$ is the *greatest* lower bound of B, we deduce that

$$\inf B \ge \sup A.$$

This concludes the proof. □

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(6) (10 points) Let f_n denote the n^{th} Fibonacci number, i.e. $f_1 = 1$, $f_2 = 1$, and $f_{n+1} = f_n + f_{n-1}$ for all natural numbers $n \geq 2$. Prove that

$$1 \leq \frac{f_{n+1}}{f_n} \leq 2$$

for all $n \in \mathbb{N}$.

We proceed by induction. Let

$$\mathcal{A} := \{n \in \mathbb{N} : 1 \leq \frac{f_{n+1}}{f_n} \leq 2\}.$$

Since $\frac{f_2}{f_1} = 1$, we see that $1 \in \mathcal{A}$. We wish to show that every natural number lives in $\mathcal{A}$.

Suppose $n \in \mathcal{A}$; this implies $1 \leq \frac{f_{n+1}}{f_n} \leq 2$, whence

$$\frac{1}{2} \leq \frac{f_n}{f_{n+1}} \leq 1. \tag{$\dagger$}$$

We wish to show that $n+1 \in \mathcal{A}$. We have:

$$\begin{aligned} \frac{f_{n+2}}{f_{n+1}} &= \frac{f_{n+1} + f_n}{f_{n+1}} \\ &= 1 + \frac{f_n}{f_{n+1}} \in [3/2, 2], \end{aligned}$$

where the last step follows from ($\dagger$). It follows that

$$1 \leq \frac{f_{n+2}}{f_{n+1}} \leq 2$$

whence $n+1 \in \mathcal{A}$. By Induction, we conclude that $\mathcal{A} = \mathbb{N}$, and thus have proved the claim. □

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Total Marks = 100 points