MATA32 – Winter 2010
Selected solutions to previous exam questions

FROM THE 2007 EXAM:

Part A

3. Note that $g''(x) = k(k-1)x^{k-2}$. Since k is between 0 and 1, $k(k-1)$ must be negative, while x^{k-2} is automatically positive whenever $x > 0$. Thus, $g''(x) < 0$ for all x, which means that $g'(x)$ is decreasing on $(0, \infty)$.

6. $c'(q) = 6q^2 + 880$, so by the fundamental theorem of calculus,

$$\begin{aligned} c(9) - c(6) &= \int_6^9 c'(q)\,dq \\ &= \int_6^9 6q^2 + 880\,dq \\ &= (2q^3 + 880q)\Big|_6^9 \\ &= 3666 \end{aligned}$$

Part B

5. Say the line is tangent to the function $y = x^2 + x$ at the point (a, b). Since the line also goes through $(2, -3)$, it has slope $\frac{b+3}{a-2}$. On the other hand, since it's tangent to the curve, the line must have slope $\frac{dy}{dx}\Big|_{x=a} = 2a + 1$. Therefore, $\dfrac{b+3}{a-2} = 2a + 1$. But also, because the point (a, b) is on the curve $y = x^2 + x$, we know that $b = a^2 + a$. So,

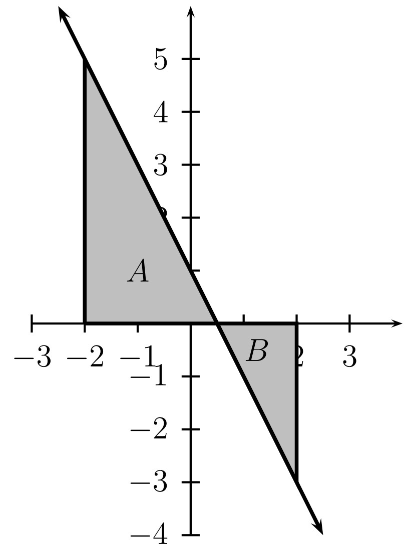
$$(2a+1)(a-2) = b + 3 = a^2 + a + 3$$

Simplifying and solving for a, we find that $a = -1$ or 5. Figuring out b for each of these possibilities, we find that $(a, b) = (-1, 0)$ or $(5, 30)$. The second one would produce a line with positive slope, so that's not the right answer. Therefore, the line goes through $(-1, 0)$ and $(2, -3)$, so its equation is $y = -(x + 1)$.

8. (a) There are two ways to do this. Probably the easier one is to use the fundamental theorem of calculus:

$$\int_{-2}^{2} (-2x + 1)\,dx = -x^2 + x\Big|_{-2}^{2} = (-4 + 2) - (-4 - 2) = 4$$

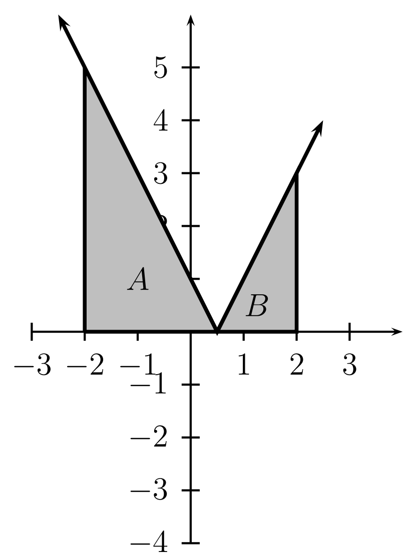
The second way, which is more instructive, is to figure out the answer directly from the definition of the integral. The picture looks like this:



Using the formula that the area of a triangle is the $\frac{1}{2}$(base)(height), we have area$(A) = \frac{25}{4}$ and area$(B) = \frac{9}{4}$. Since B is below the x-axis, we count this area negatively, i.e.

$$\int_{-2}^{2} (-2x+1)\,dx = \text{area}(A) - \text{area}(B) = 4$$

(c) This time, it's best to solve using the definition of the integral. The picture is the same as above, except that triangle B will be flipped over the x-axis:



So,

$$\int_{-2}^{2} \left|-2x+1\right|\,dx = \text{area}(A) + \text{area}(B) = 8.5$$

FROM THE 2008 EXAM:

<u>Part A</u>

2. By the fundamental theorem of calculus,

$$\int_0^1 y'\,dx = y(1) - y(0) = 2 - y(0)$$

The left hand side of the above is

$$\int_0^1 (3\sqrt{x} - 1)\,dx = \left(2x^{3/2} - x\right)\Big|_0^1 = 1$$

It follows that $2 - y(0) = 1$, so $y(0) = 1$.

6. Let $u = 3x^2 + 1$, so that $du = 6x\,dx$. Also, when $x = 0$ we have $u = 1$, and when $x = 1$ we have $u = 4$. Therefore,

$$\int_0^1 \frac{3x}{3x^2+1}\,dx = \int_1^4 \frac{\frac{1}{2}\,du}{u} = \frac{1}{2}\int_1^4 \frac{1}{u}\,du = \frac{1}{2}\ln u\Big|_1^4 = \frac{1}{2}\ln 4 = \ln 2$$

<u>Part B</u>

1. Implicitly differentiating, we find that

$$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

Solving this for $\frac{dy}{dx}$ gives

$$\frac{dy}{dx} = -\frac{2x+y}{x+2y}$$

We want to find for which points (x, y) on the original curve does the tangent line have a slope of 1. In other words, we want to find all (x, y) such that

$$x^2 + xy + y^2 = 9 \tag{1}$$

and

$$\frac{dy}{dx} = -\frac{2x+y}{x+2y} = 1. \tag{2}$$

Simplifying (2) yields $y = -x$. Plugging this into (1) leads to $x^2 = 9$, from which we deduce that $x = \pm 3$. We conclude that at both $(3, -3)$ and $(-3, 3)$ the slope of the tangent line is 1.

5. (a) The limit definition of the derivative of $f(x)$ at $x = a$ is

$$f'(a) = \lim_{h\to 0}\frac{f(a+h) - f(a)}{h}$$

So, $f(x)$ is differentiable at $x = a$ if the above limit exists. For example, if $f(x)$ has a 'corner' at $x = a$, then the limit from the left won't equal the limit from the right, and so the above limit would not exist.

(c) The fundamental theorem of calculus states that

$$\int_a^b f'(x)\,dx = f(x)\Big|_a^b = f(b) - f(a).$$

Note that this is a *theorem*: the definition of the integral is quite different, and seemingly unrelated to this (recall that it's the area between $f(x)$ and the x-axis in the region bounded by the vertical lines $x = a$ and $x = b$).

9. (a) Recall that profit is $P(x) = R - C$, where R is the revenue and C is the cost. Since q units per week are produced at a price of x dollars per unit, the revenue (in dollars per week) is simply qx. Similarly, since the cost per unit is m dollars, and q of them are produced per week, the total cost (in dollars per week) is qm. Therefore,

$$P(x) = qx - qm = (x - m)q = 75 + 80(5m - x)(x - m) = -80x^2 + 480mx + 75 - 400m^2$$

(b) We want to maximize the function $P(x)$ on the interval $x \in (m, 5m)$. First step: differentiate $P(x)$ and set it equal to 0.

$$P'(x) = -160x + 480m = 0$$

so this happens at $x = 3m$. (Note that this value of x is inside the allowed interval.) Does $P(x)$ have a min or a max here? The second derivative is easy to compute: $P''(x) = -160$. It follows that $P''(3m) = -160 < 0$, so $P(x)$ must have a relative max at $x = 3m$. It remains only to determine whether this relative max is also an *absolute* max on the interval $(m, 5m)$. To do this, we evaluate $P(x)$ at $x = 3m$ and at the endpoints, and compare. Plugging in these three values of x into the expression for $P(x)$ we found in part (a) gives

$$P(3m) = 320m^2 + 75 \qquad P(m) = 75 \qquad P(5m) = 75$$

Since $P(x)$ is clearly largest at $x = 3m$, this must indeed be an absolute maximum on the interval.

(c) In (b) we figured out that the maximum weekly profit is $P(3m) = 320m^2 + 75$. For this to equal \$3455, m would have to be \$3.25.