L'HÔPITAL'S RULE

LEO GOLDMAKHER

L'Hôpital's rule, which appears in many guises and is a powerful tool for calculating limits, is somewhat cumbersome to prove (see Spivak's treatment). Here I write down a short argument which is not rigorous but captures the essence of the proof; hopefully, this will serve as a roadmap for reading the formal proof without getting lost in the details. As you read the false "proof" below, try to determine which steps are and are not legit.

Theorem (L'Hôpital's Rule)**.** *Suppose* $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$*. Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

assuming the right hand side exists.

"Proof". It is tempting to assume that $f(a)$ and $g(a)$ are 0. In fact, the statement of the theorem does not even presuppose that these quantities are defined! To rectify this, we work instead with the two functions

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases} \qquad \text{and} \qquad \widetilde{g}(x) = \begin{cases} g(x) & \text{if } x \neq a \\ 0 & \text{if } x = a. \end{cases}$$

Note that both $\widetilde{f}$ and $\widetilde{g}$ are continuous at a.

By the Mean Value Theorem, for any $x > a$ there is a $c_x \in (a, x)$ such that

$$\widetilde{f}'(c_x) = \frac{\widetilde{f}(x) - \widetilde{f}(a)}{x - a}.$$

Since $\widetilde{f}(a) = 0$, it follows that $\widetilde{f}(x) = (x - a)\widetilde{f}'(c_x)$. Similarly, $\forall x > a$, $\exists d_x \in (a, x)$ such that $\widetilde{g}(x) = (x - a)\widetilde{g}'(d_x)$. It follows that

$$\begin{aligned}
\lim_{x\to a} \frac{f(x)}{g(x)} &= \lim_{x\to a} \frac{\widetilde{f}(x)}{\widetilde{g}(x)} \\
&= \lim_{x\to a} \frac{(x-a)\widetilde{f}'(c_x)}{(x-a)\widetilde{g}'(d_x)} && \text{by the MVT, as described above} \\
&= \lim_{x\to a} \frac{\widetilde{f}'(c_x)}{\widetilde{g}'(d_x)} \\
&= \lim_{y\to a} \frac{\widetilde{f}'(y)}{\widetilde{g}'(y)} && \text{since as } x \to a, \text{ both } c_x \to a \text{ and } d_x \to a \\
&= \lim_{y\to a} \frac{f'(y)}{g'(y)}.
\end{aligned}$$

□

I urge you to go through the proof carefully to determine which steps can be rigorously justified without too much effort, and which are less obvious how to justify. Here are some questions to start you off:

(1) What allows us to apply the MVT? After all, the MVT has various hypotheses which I never mentioned during the course of the proof. Are they actually satisfied?

(2) In the chain of equalities at the end of the proof, are the first and last ones justified? (i.e. is it OK to pass between $\widetilde{f}$ and f, or $\widetilde{f}'$ and f', in this way?)

(3) Is it really true that

$$\lim_{x\to a} \frac{\widetilde{f}'(c_x)}{\widetilde{g}'(d_x)} = \lim_{y\to a} \frac{\widetilde{f}'(y)}{\widetilde{g}'(y)}?$$

Why?

HTTP://WEB.WILLIAMS.EDU/MATHEMATICS/LG5/

E-mail address: Leo.Goldmakher@williams.edu