UNIVERSITY OF TORONTO SCARBOROUGH

MATA37H3 : Calculus for Mathematical Sciences II

MIDTERM EXAMINATION # 1

January 30, 2012

Duration – 2 hours
Aids: none

NAME (PRINT): SOLUTION KEY
Last/Surname First/Given Name

STUDENT NO: ____________________

TUTORIAL: ____________________
Tutorial section (Number or Schedule) Name of TA

Qn. #	Value	Score
1	12	
2	10	
3	18	
4	20	
5	20	
6	20	
Total	100	

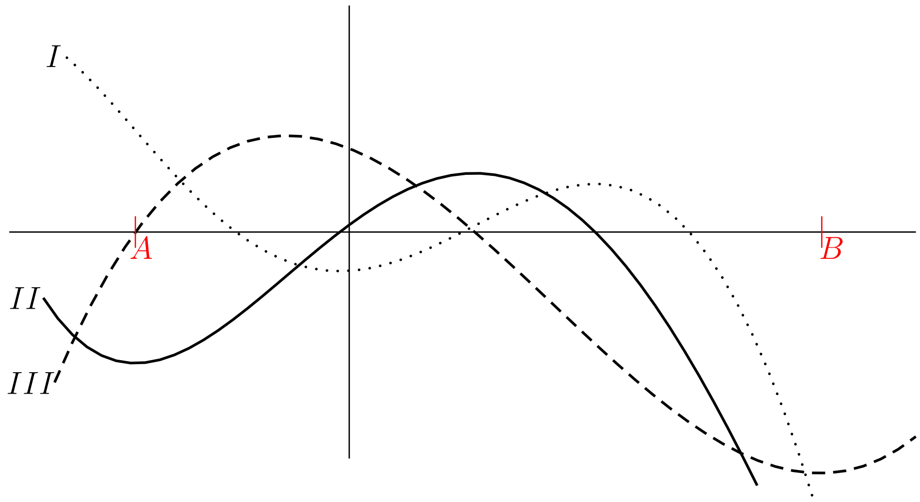
TOTAL: ____________

Please read the following statement and sign below:

I understand that any breach of academic integrity is a violation of The Code of Behaviour on Academic Matters. By signing below, I pledge to abide by the Code.

SIGNATURE:____________________

(1) (12 points) On the axes below are graphed f, f', and f''. Determine which is which, and justify your response with a brief explanation.



I : f

II : f'

III : f''

Explanation:

The derivative of function III is negative to the left of $x = B$ and positive to the right; since neither I nor II behave this way, III must be f''. This tells us that f'' is 0 at $x = A$. The derivative of I is negative at $x = A$, while the derivative of II is zero; this shows that II must be f'. This in turn implies that I must be f.

continued on page 3

(2) (10 points) Suppose f is a function with $f(2) = 1$ and $f'(2) = -5$. Use this information to approximate $f(2.1)$. Justify your answer.

The tangent line to f at 2 is $L(x) = -5(x-2)+1$. Since 2.1 is fairly close to 2, it's reasonable to expect that $L(2.1)$ will be a good approximation to $f(2.1)$. Thus, we conclude that $f(2.1) \approx 0.5$

continued on page 4

(3) For this problem, assume that f and g are functions such that both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ exist.

(a) (10 points) Prove that if $f(x) \leq g(x)$ for all x, then $\lim_{x\to\infty} f(x) \leq \lim_{x\to\infty} g(x)$.

Let $L_f := \lim_{x\to\infty} f(x)$ and $L_g := \lim_{x\to\infty} g(x)$, and suppose $L_f - L_g = \epsilon > 0$. By definition of the limit, there exists an $x_f \in \mathbb{R}$ such that $|f(x) - L_f| < \epsilon/10$ for all $x > x_f$, and an $x_g \in \mathbb{R}$ such that $|g(x) - L_g| < \epsilon/10$ for all $x > x_g$. Set $x_0 = \max\{x_f, x_g\}$. Then for any $x > x_0$ we have

$$f(x) > L_f - \epsilon/10 > L_g + \epsilon/10 > g(x).$$

But this contradicts our hypothesis that $f(x) \leq g(x)$.

(b) (8 points) Does (a) hold if we replace all instances of $\leq$ by $<$? In other words, if $f(x) < g(x)$ for all x, must it be true that $\lim_{x\to\infty} f(x) < \lim_{x\to\infty} g(x)$? If so, prove it. If not, give an example.

(a) does not hold with strict inequalities. For example, $1 - 1/(1+x^2) < 1$ for all $x \in \mathbb{R}$, while

$$\lim_{x\to\infty} 1 - \frac{1}{1+x^2} = 1 = \lim_{x\to\infty} 1.$$

continued on page 5

(4) (20 points) Suppose f is a function satisfying $f(0) = f'(0) = 0$. Prove that there exists an open interval I such that $0 \in I$ and $|f(x)| \leq \frac{|x|}{100}$ for all $x \in I$.

By definition of the derivative, we have

$$0 = f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h}.$$

By definition of the limit, there must exist $\delta > 0$ such that

(*) $$\left| \frac{f(x)}{x} \right| < \frac{1}{100}$$

for all x satisfying $0 < |x| < \delta$. Let I denote the interval $(-\delta, \delta)$. From (*), any nonzero $x \in I$ satisfies $|f(x)| < |x|/100$. Moreover, we have $f(0) = 0$. The claim follows.

continued on page 6

(5) (20 points) Prove that $\cos x$ is continuous on $\mathbb{R}$. You may use any result proved in class.

We proved in class that $\cos x$ is differentiable everywhere. We also proved that if a function is differentiable at a point, it must be continuous at that point. It follows that $\cos x$ is continuous everywhere.

continued on page 7

(6) Let $f(x) = \cos x$ where x is in *degrees*.

(a) (10 points) Is $f'(90)$ positive, negative, or zero? Justify your answer.

For this problem, denote by $\cos_d y$ the function which gives the cosine of y degrees, and by $\cos_r y$ the function which gives the cosine of y radians, and similarly for the sine function. We have $f(x) := \cos_d(x)$; set $g(x) := \cos_r(x)$. Note that $f(x) = g\left(\frac{\pi}{180}x\right)$. Also, from class, we know that $g'(x) = -\sin_r(x)$.

By definition, we have

$$f'(90) = \lim_{h\to 0} \frac{f(90+h) - f(90)}{h} = \lim_{h\to 0} \frac{f(90+h)}{h}.$$

Now, $f(90+h) = g(\pi/2 + \pi h/180)$. It follows that

$$\begin{aligned} f'(90) &= \lim_{h\to 0} \frac{g\left(\frac{\pi}{2} + \frac{\pi}{180}h\right)}{h} \\ &= \frac{\pi}{180} \lim_{h\to 0} \frac{g\left(\frac{\pi}{2} + \frac{\pi}{180}h\right)}{\frac{\pi}{180}h} \\ &= \frac{\pi}{180} \lim_{k\to 0} \frac{g\left(\frac{\pi}{2} + k\right)}{k} \\ &= \frac{\pi}{180} g'(\frac{\pi}{2}) \\ &= \frac{\pi}{180}\Big(-sin_r(\frac{\pi}{2})\Big) \\ &= -\frac{\pi}{180} \end{aligned}$$

where the passage from the limit as $h \to 0$ to the limit as $k \to 0$ is justified by the lemma proved in lecture: if c is a constant and G is a function such that $\lim_{h\to 0} G(h)$ exists, then

$$\lim_{h\to 0} G(ch) = \lim_{k\to 0} G(k).$$

Finally, we conclude that $f'(90) < 0$.

(b) (10 points) Is $|f'(90)|$ larger, smaller, or equal to $1/2$? Justify your answer.

From above, $|f'(90)| = \frac{\pi}{180} < 1/2$.

continued on page 8

Page intentionally left blank for scratch work.

Total Marks = 100 points