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MAT1302 / APM 461: ADDITIVE COMBINATORICS

Problem Set 1 – due Friday, February 14

INSTRUCTIONS:
Please turn the assignment in at the *start* of the lecture on the due date. Please print and attach this page as the first page of your submitted problem set.

PROBLEM	**MARK**
1.1	
1.2	
1.3	
1.4	
1.5	
1.6	
1.7	
1.8	
1.9	
1.10	
Total	

Notation:

- $f(x) = O\big(g(x)\big)$ means that $g(x) > 0$ and $\left|\frac{f(x)}{g(x)}\right|$ is bounded for all reasonable values of x.
- $f(x) \ll g(x)$ means precisely the same thing as $f(x) = O\big(g(x)\big)$.
- $f(x) \gg g(x)$ means $g(x) \ll f(x)$.
- $f(x) = o\big(g(x)\big)$ means $\frac{f(x)}{g(x)} \to 0$ as $x \to \infty$.
- $f(x) \asymp g(x)$ means $f(x) \ll g(x)$ and $g(x) \ll f(x)$.
- $f(x) \sim g(x)$ means $\frac{f(x)}{g(x)} \to 1$ as $x \to \infty$.
- $f(x) = g(x) + O\big(h(x)\big)$ means $f(x) - g(x) = O\big(h(x)\big)$.

Problem Set 1

1.1 In this exercise, you will explore the connection between $\sim$ and $o(\cdot)$.

(a) Prove that $f(x) \sim g(x)$ if and only if $f(x) = \big(1 + o(1)\big)g(x)$.

(b) Prove that for all nice functions $f(x)$ and $g(x)$, $f(x) \sim g(x)$ if and only if $\log f(x) = \log g(x) + o(1)$. Give an appropriate interpretation of 'nice'.

1.2 In each of the following, determine which of the following relations hold (possibly more than one):

$$f(x) \sim g(x), \quad f(x) \ll g(x), \quad f(x) \gg g(x), \quad f(x) \asymp g(x), \quad f(x) = o\big(g(x)\big), \quad g(x) = o\big(f(x)\big).$$

Justify your responses.

(a) $f(x) = 5x^3 - 1000x + 2$ and $g(x) = x^3 - 0.01$

(b) $f(x) = 1000x$ and $g(x) = x^3$

(c) $f(x) = x^{1000}$ and $g(x) = 2^x$

(d) $f(x) = \log x$ and $g(x) = x^{0.01}$

(e) $f(x) = x^{\log x}$ and $g(x) = (\log x)^x$

(f) $f(x) = (\log x)^{\log x}$ and $g(x) = x^{\log \log x}$.

1.3 The purpose of this exercise is to introduce a simple but powerful technique called *partial summation.*

(a) Given any (Riemann) integrable function f, any sequence of complex numbers a_n, and any $x \geq 1$, prove that

$$\int_1^x f'(t)\bigg(\sum_{n \leq t} a_n\bigg)\,dt = \sum_{n \leq x} a_n\Big(f(x) - f(n)\Big).$$

[*Hint: Build up your intuition by proving the simple case $x = 2$. Then try $x = 3$. Then extend to all real $x \geq 1$.*]

(b) *Partial summation* is the formula

$$\sum_{n \leq x} a_n f(n) = \int_{1^-}^x f(t)\,d\bigg(\sum_{n \leq t} a_n\bigg), \tag{*}$$

where 1^- is shorthand for $1 - \epsilon$ with $\epsilon \to 0^+$, and the integral should be evaluated by integration by parts:

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du.$$

Prove (*).

[**Comment**: Partial summation has a nice heuristic interpretation – not helpful for solving this problem, but a good mnemonic for remembering the formula (*). We start with t slightly below 1. As t increases, $\sum_{n \leq t} a_n$ doesn't change until t crosses the value 1; at that moment, $\sum_{n \leq t} a_n$ changes by a_1, and the contribution to the integral is $a_1 f(1)$. Next, as t increases from 1 to 2, $\sum_{n\le t} a_n$ sees no change; as soon as t crosses 2, the sum changes by a_2, and the contribution to the integral is $a_2 f(2)$. Continuing the process suggests (*).]

1.4 Let $[x]$ denote the largest integer less than or equal to x, i.e.

$$[x] := \sup\Big\{n \in \mathbb{Z} : n \le x\Big\}.$$

This is typically called the *floor* or *integer part* of x. The *fractional part* of x is defined $\{x\} := x - [x]$.

(a) Prove that for any positive integer N,

$$\log N! = N \log N - N + \frac{1}{2}\log N + 1 + \int_1^N \left(\{t\} - \frac{1}{2}\right)\frac{dt}{t}.$$

[*Hint: Show that* $\log N! = \int_{1^-}^N \log t \, d[t]$ *and integrate by parts.*]

(b) Prove that

$$\int_1^N \left(\{t\} - \frac{1}{2}\right)\frac{dt}{t} = O(1).$$

(c) Prove that $N! \asymp \sqrt{N}\left(\frac{N}{e}\right)^N$.

1.5 Euler discovered the following identity, valid for any nonzero x (measured in radians):

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right). \tag{†}$$

(Recall that $\prod a_n$ denotes the product of all the a_n's.) Euler used this to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

In this exercise, you will derive a different consequence: Stirling's formula.

(a) Using Euler's identity (†) or otherwise, prove that

$$\frac{(2N)!!}{(2N-1)!!} \sim \sqrt{\pi N}.$$

Here $(2N)!!$ denotes the product of all positive even numbers $\le 2N$, and $(2N-1)!!$ denotes the product of all positive odd number $\le 2N-1$.

(b) Prove that

$$(2N)!! = 2^N N!$$

and

$$(2N-1)!! = \frac{(2N)!}{2^N N!}$$

(c) Prove Stirling's formula: $N! \sim \sqrt{2\pi N}\left(\frac{N}{e}\right)^N$. [*Hint: First prove that there exists a constant C such that* $N! \sim C\sqrt{N}\left(\frac{N}{e}\right)^N$. *Now use parts (a) and (b) to show that C must equal* $\sqrt{2\pi}$.]

1.6 Having warmed up with Stirling's formula, we now apply partial summation to study primes.

(a) Recall from lecture the Prime Number Theorem, which asserts that

$$\pi(x) \sim \int_2^x \frac{dt}{\log t}$$

(where $\pi(x)$ denotes the number of primes $\le x$). Use this to prove that

$$\pi(x) \sim \frac{x}{\log x}.$$

[**Comment**: While this formula is more explicit than the previous one, and is occasionally easier to use, it is a much worse approximation to $\pi(x)$.]

(b) One precise version of the Prime Number Theorem states that

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^{10}}\right).$$

Use this to prove that

$$\sum_{p \le x} \frac{1}{p} = \log\log x + O(1),$$

a fact we used in lecture. [*Hint: Use partial summation to show that* $\sum_{p\le x} f(p) = \int_{2^-}^x f(t)\, d\pi(t)$. *Now integrate by parts and simplify.*]

1.7 Suppose $A \subseteq \mathbb{Z}$ is finite. Prove that $|A + A| \ge 2|A| - 1$, and that equality holds if and only if A is an arithmetic progression.

1.8 Given a group G, suppose H is a finite *subset* of G which is closed under the binary operation of G. (As usual, we will denote this operation as a product.) The goal of this exercise is to prove that H must be a subgroup of G.

(a) Carefully explain why associativity holds in H.

(b) Prove that the identity $e \in H$. [*Hint*: Let $n = |H|$, the order of H. For any $a \in H$ which is not the identity, consider the set $S = \{a, a^2, a^3, \ldots, a^{n+1}\}$. Why must S be a subset of H? Why must two elements of S be equal to each other? Deduce that $e \in S$, and therefore, $e \in H$.]

(c) Prove that for all $a \in H$, we have $a^{-1} \in H$ (where a^{-1} is the inverse of a in G).

1.9 Exercise 1 from Lecture 1

1.10 Exercise 2 from Lecture 1