UNIVERSITY OF TORONTO SCARBOROUGH

MATB43 : Introduction to Analysis

FINAL EXAMINATION

April 25, 2013

Duration – 3 hours
Aids: none

NAME (PRINT): KEY
Last/Surname First/Given Name (and nickname)

STUDENT NO: ____________________

Qn. #	Value	Score
1	18	
2	12	
3	10	
4	30	
5	30	
Total	100	

TOTAL: ____________

Please read the following statement and sign below:

I understand that any breach of academic integrity is a violation of The Code of Behaviour on Academic Matters. By signing below, I pledge to abide by the Code.

SIGNATURE:____________________

(1) In all parts of this problem, $f : \left(-\frac{1}{2}, \frac{1}{2}\right) \to \mathbb{R}$ is defined by

$$f(x) = \frac{x}{\frac{1}{2} - |x|}.$$

(a) (6 points) Prove that f is a bijection.

Injectivity: Suppose $f(x) = f(y)$. We wish to show that $x = y$. We consider two cases:

Case (i) $x \geq 0$. Then $f(x) \geq 0$, whence $f(y) \geq 0$, whence $y \geq 0$. Since $f(x) = f(y)$, we have

$$\frac{x}{\frac{1}{2} - x} = \frac{y}{\frac{1}{2} - y}.$$

Simplifying this yields $x = y$.

Case (ii) $x < 0$. Then $f(x) < 0$, whence $f(y) < 0$, whence $y < 0$. Since $f(x) = f(y)$, we have

$$\frac{x}{\frac{1}{2} + x} = \frac{y}{\frac{1}{2} + y}.$$

Simplifying this yields $x = y$.

Surjectivity: It is easily verified that for any $\alpha \in \mathbb{R}$, we have

$$f\left(\frac{\alpha}{2(1 + |\alpha|)}\right) = \alpha. \qquad (\dagger)$$

(b) (6 points) Given $x, y \in (-\frac{1}{2}, \frac{1}{2})$, define $d(x, y) := |f(x) - f(y)|$. Prove that d is a metric on $(-\frac{1}{2}, \frac{1}{2})$.

We must verify that the three properties of a metric are satisfied. First, we have $d(x, x) = 0$. Moreover, if $d(x, y) = 0$, then $f(x) = f(y)$; by injectivity, $x = y$. Thus, we've proved that $d(x, y) = 0$ iff $x = y$.

Next, $d(x, y) = |f(x) - f(y)| = |f(y) - f(x)| = d(y, x)$.

Finally, for any x, y, z we have

$$d(x, y) + d(y, z) = |f(x) - f(y)| + |f(y) - f(z)| \geq |f(x) - f(z)| = d(x, z),$$

by triangle inequality for absolute value.

(c) (6 points) Give an explicit description of $\mathcal{N}_{50}\left(\frac{1}{4}\right)$, i.e. the neighbourhood of radius 50 around the point $1/4$. (By "explicit description" I mean tell me all the elements of this neighbourhood.)

By definition,

$$\mathcal{N}_{50}\left(\frac{1}{4}\right) = \left\{x \in \left(-\frac{1}{2}, \frac{1}{2}\right) \ : \ d\left(x, \frac{1}{4}\right) < 50\right\}.$$

Now, $d(x, \frac{1}{4}) = |f(x) - f(1/4)| = |f(x) - 1|$, so $x \in \mathcal{N}_{50}\left(\frac{1}{4}\right)$ if and only if

$$-49 < f(x) < 51. \qquad (\ddagger)$$

f is easily seen to be increasing on $\left(-\frac{1}{2}, \frac{1}{2}\right)$, so ($\ddagger$) is equivalent to

$$f^{-1}(-49) < x < f^{-1}(51).$$

By ($\dagger$) in part (a), we see that

$$f^{-1}(\alpha) = \frac{\alpha}{2(1 + |\alpha|)}.$$

It follows that

$$\mathcal{N}_{50}\left(\frac{1}{4}\right) = \left(-\frac{49}{100}, \frac{51}{104}\right).$$

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(2) Consider the metric space $\mathbb{R}^2$ under the British Rail metric, i.e.

$$d(x,y) := \begin{cases} 0 & \text{if } x = y \\ |x| + |y| & \text{if } x \neq y, \end{cases}$$

where $|x|$ denotes the usual absolute value in $\mathbb{R}^2$ (i.e. the distance from the origin). Let

$$S := \{(a,b) \in \mathbb{R}^2 : (a-2)^2 + (b-2)^2 < 1\}.$$

(a) (6 points) Is S open in the metric space $(\mathbb{R}^2, d)$? Prove it or carefully explain why not.

Yes, S is open. To see this, we first prove the following:

Lemma. *Given $p \in \mathbb{R}^2$. Then $\mathcal{N}_{\frac{1}{2}|p|}(p) = \{p\}$.*

Proof. Since $\frac{1}{2}|p| \geq 0$ for all $p \in \mathbb{R}^2$, and $d(p,p) = 0$, we see that $p \in \mathcal{N}_{\frac{1}{2}|p|}(p)$ for all $p \in \mathbb{R}^2$. It therefore suffices to prove that this neighbourhood contains no points aside from p.

Now, suppose $x \neq p$. Then

$$d(x,p) = |x| + |p| \geq \frac{1}{2}|p|,$$

from which we deduce that $x \notin \mathcal{N}_{\frac{1}{2}|p|}(p)$. $\square$

We now prove that S is open. First observe that given any $p \neq (0,0)$, we have $\frac{1}{2}|p| > 0$, whence (by the lemma) there exists a neighbourhood around p which consists entirely of p. It follows that for any $p \neq (0,0)$, the singleton set $\{p\}$ is open. Since arbitrary unions of open sets are open, we conclude that S is open.

(b) (6 points) Is S closed in the metric space $(\mathbb{R}^2, d)$? Prove it or carefully explain why not.

Yes, S is closed. It suffices to show that the complement $\mathbb{R}^2 - S$ is open. Note that

$$\mathbb{R}^2 - S = \left\{x \in \mathbb{R}^2 : |x| < \frac{1}{10}\right\} \cup \left\{x \in \mathbb{R}^2 - S : x \neq (0,0)\right\}. \qquad (*)$$

We now show each of the sets on the right hand side is open; this will prove that $\mathbb{R}^2 - S$ is open, since unions of open sets are open.

First, we have

$$\begin{aligned}\left\{x \in \mathbb{R}^2 : |x| < \frac{1}{10}\right\} &= \left\{x \in \mathbb{R}^2 : d\big(x,(0,0)\big) < \frac{1}{10}\right\} \\ &= \mathcal{N}_{\frac{1}{10}}\big((0,0)\big).\end{aligned}$$

Since we proved in class that in any metric space, $\mathcal{N}_r(p)$ is open for any $r > 0$, we conclude that $\left\{x \in \mathbb{R}^2 : |x| < \frac{1}{10}\right\}$ is open.

Next, observe that

$$\left\{x \in \mathbb{R}^2 - S : x \neq (0,0)\right\} = \bigcup_{\substack{x \in \mathbb{R}^2 - S \\ x \neq (0,0)}} \{x\}.$$

Just as in part (a), it follows that $\left\{x \in \mathbb{R}^2 - S : x \neq (0,0)\right\}$ must be open, since it is the union of open sets.

We have thus shown that both sets on the right hand side of (*) are open. We conclude that $\mathbb{R}^2 - S$ is open, and hence, that S is closed.

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(3) (a) (5 points) Let

$$a_n := \begin{cases} 1/2 & \text{if } n = 2^k \text{ for some } k \in \mathbb{N} \\ \frac{1}{n^2} & \text{otherwise.} \end{cases}$$

Does $\displaystyle\sum_{n=1}^{\infty} a_n$ converge? Justify your answer with a proof.

Note that (a_n) does not tend to 0 as $n \to \infty$, since for all N, there exists $n > N$ such that $|a_n - 0| = \frac{1}{2}$. It follows that the series cannot converge, since for any convergent series, the terms must tend to 0.

(b) (5 points) Let $b_n := \frac{2^n}{n!}$ for all $n \in \mathbb{N}$. Does $\displaystyle\sum_{n=1}^{\infty} b_n$ converge? Justify your answer with a proof.

We apply the ratio test:

$$\begin{aligned} \lim_{n\to\infty} \frac{b_{n+1}}{b_n} &= \lim_{n\to\infty} \frac{n!}{(n+1)!} \cdot \frac{2^{n+1}}{2^n} \\ &= \lim_{n\to\infty} \frac{2}{n+1} \\ &= 0. \end{aligned}$$

(The limit tends to 0 because, for any $\epsilon > 0$, we have $\left|\frac{2}{n+1} - 0\right| < \epsilon$ whenever $n > \frac{2}{\epsilon} - 1$.)

In particular, we see that $\lim_{n\to\infty} \frac{b_{n+1}}{b_n} < 1$. The ratio test implies that the series converges.

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(4) (30 points) Prove Hölder's inequality: for any $p, q > 1$ such that $1/p + 1/q = 1$, for any sequences $(a_n), (b_n)$ of real numbers, for any $N \in \mathbb{N}$, we have

$$\left| \sum_{n=1}^{N} a_n b_n \right| \leq \left(\sum_{n=1}^{N} |a_n|^p \right)^{1/p} \left(\sum_{n=1}^{N} |b_n|^q \right)^{1/q}.$$

See lecture summaries.

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(5) (a) (10 points) Prove the Monotone Subsequence theorem: every sequence of real numbers contains a monotone subsequence.

See lecture summaries.

(b) (10 points) Prove the Bolzano-Weierstrass theorem: every bounded sequence has a convergent subsequence.

See lecture summaries.

(c) (10 points) Prove the Cauchy criterion: a sequence of real numbers (a_n) converges if and only if (a_n) is Cauchy.

See lecture summaries.

Total Marks = 100 points