LECTURE 14: SUMMARY

We began with the following important result:

Theorem 1 (Monotone Convergence Theorem)**.** *Suppose (a_n) is a monotone sequence. Then (a_n) converges iff (a_n) is bounded.*

Before proving this, we need to define some of the words appearing in the statement of the theorem.

Definition. *A sequence (a_n) is* monotonically increasing *iff $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$; it is* monotonically decreasing *iff $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is said to be* monotone *iff it is either monotonically increasing or monotonically decreasing.*

Definition. *A sequence (a_n) is* bounded above *iff $\exists M \in \mathbb{R}$ such that $a_n \leq M$ for all $n \in \mathbb{N}$; it is* bounded below *iff $\exists m \in \mathbb{R}$ such that $a_n \geq m$ for all $n \in \mathbb{N}$. A sequence is said to be* bounded *iff it is both bounded above and below.*

We can now prove the Monotone Convergence Theorem (henceforth called the MCT).[1] Although the proof is simple, this theorem is extremely useful, as we shall see in the next few weeks.

Proof of MCT. As usual, we prove the two implications of the theorem separately.

($\Longrightarrow$) Suppose (a_n) converges to A. Then there exists N such that $a_n \in (A-1, A+1)$ for all $n > N$, whence the set $\{a_n : n > N\}$ is bounded. The range of the beginning of the sequence, $\{a_n : n \leq N\}$, is finite and hence also bounded. We conclude that the entire set $\{a_n\}$ is bounded, as claimed. Sean pointed out that we didn't use the monotonicity of the sequence anywhere in this argument.

($\Longrightarrow$) We know the sequence is monotone; for convenience, let's suppose it's monotonically increasing (if it's decreasing, a similar argument will work). Suppose (a_n) is bounded. Then the set $\{a_n\}$ is bounded above, whence $A := \sup\{a_n\}$ exists. I claim that

$$\lim_{n\to\infty} a_n = A.$$

To see this, suppose $\epsilon > 0$. By definition of supremum, there must exist an $N \in \mathbb{N}$ such that $A - a_N < \epsilon$. Since the sequence is monotonically increasing, $a_n \geq a_N$ for all $n > N$, whence for all such n

$$|a_n - A| = A - a_n \leq A - a_N < \epsilon. \qquad \square$$

Date: February 28th, 2013.

[1] This name is more frequently used to refer to a theorem about sequences of measurable functions; you will encounter this result in a course on real analysis and measure theory.

Next, we discussed a very important example of an infinite series, called the harmonic series:

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

Does it converge or diverge? Simple numerical experiments indicate that the partial sums

$$S_N := \sum_{n \le N} \frac{1}{n}$$

converge (for example, $S_{1000000} < 15$). By comparing S_N to a Riemann sum for $\int_1^N \frac{dx}{x}$, however, we proved that $S_N \ge \log N$ for all N (in fact, we proved an upper bound on S_N of similar magnitude). It follows that the sequence (S_N) is unbounded, and thus diverges.

By playing around with this idea more, we were able to prove the following:

Theorem 2. *Suppose $f(x) \ge 0$ and decreasing for all $x \ge 1$. Then for all $N \in \mathbb{N}$,*

$$f(N) \le \sum_{n \le N} f(n) - \int_1^N f(x)\, dx \le f(1).$$

Roughly, Theorem 2 says that for non-negative monotonic functions f,

$$\sum_{n \le N} f(n) \approx \int_1^N f(x)\, dx.$$

Proof. For all $n \ge 2$ we have

$$f(n) \le \int_{n-1}^{n} f(x)\, dx \le f(n-1).$$

Summing this for $n = 2, 3, \ldots, N$ gives the result. □

One immediate consequence of Theorem 2 is a uesful test for convergence of some infinite series:

Corollary 3 (The Integral Test)**.** *Suppose $f(x) \ge 0$ and decreasing for all $x \ge 1$, and that $f(x) \to 0$ as $x \to \infty$. Then $\displaystyle\sum_{n=1}^{\infty} f(n)$ converges if and only if $\displaystyle\int_1^{\infty} f(x)\, dx$ converges.*

Proof. Let

$$S_N := \sum_{n \le N} f(n) \qquad \text{and} \qquad I_N := \int_1^N f(x)\, dx.$$

Both (S_N) and (I_N) are monotonically increasing. The improper integral converges iff the sequence (I_N) does, which (by the MCT) happens iff (I_N) is bounded. Now by Theorem 2, (I_N) is bounded iff (S_N) is bounded, which (again by the MCT) occurs iff (S_N) converges. □

One of the hypotheses of the integral test is that the terms of the series tend to 0. This is a general requirement for a sequence to converge, as we now show:

Theorem 4. *If* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

Proof. Given $\epsilon > 0$. We are given that the partial sums (S_N) converge, say, to S, whence there exists N such that

$$|S - S_n| < \epsilon/2$$

for all $n > N$. But then for any $n > N + 1$ it follows that

$$|a_n| = |S_n - S_{n-1}| = |(S_n - S) + (S - S_{n-1})| \leq |S_n - S| + |S - S_{n-1}| < \epsilon.$$

This concludes the proof. $\square$